

Poincaré and $sl(2)$ algebras of order 3

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February 7, 2008

Abstract

In this paper we initiate a general classification for Lie algebras of order 3 and we give all Lie algebras of order 3 based on $sl(2, \mathbb{C})$ and $iso(1, 3)$ the Poincaré algebra in four-dimensions. We then set the basis of the theory of the deformations (in the Gerstenhaber sense) and contractions for Lie algebras of order 3.

Pacs 02.20Sv, 11.30.Ly

keywords Contractions, generalized Inönü-Wigner, Lie algebras

1 Introduction and motivation

The concept of symmetry, and its associated algebraic structures, is central in the understanding of the properties of physical systems. This means, in particular, that a better comprehension of the laws of physics may be achieved by an identification of the possible mathematical structures as well as their classification. For instance, the properties of elementary particles and their interactions are very well understood within Lie algebras. Moreover, the discovery of supersymmetry gave rise to the concept of Lie superalgebras which becomes central in theoretical physics and mathematics. Of course not all the mathematical structures would be relevant in physics. For instance, they are constraint by the principle of quantum mechanics and relativity. This was synthesized in two no-go theorems which restrict drastically the possible Lie algebras [2] and Lie superalgebras [15] one is able to consider in physics.

But it turns out that Lie (super)algebras are not the only allowed structures one is able to consider. Several attempts to construct models based on different algebras were proposed. Here we focus on one of the possible extensions called fractional supersymmetry [3, 4, 5, 6, 18, 19, 20, 21, 22, 24, 25, 28, 29] together with its associated underlying algebraic structure named F -Lie algebra or Lie algebra of order F [26, 27] (note that a different approach has been proposed in [17]). Lie algebras of order F lead to new models of symmetry of space-time *i.e.* lead to some new non-trivial extensions of the Poincaré algebra, which involves F -ary ($F \geq 3$) relations instead of the usual quadratic ones [20, 21, 22, 29]. These new structures can be seen as a possible generalization of Lie (super)algebras. An F -Lie algebra admits a \mathbb{Z}_F -gradation, the zero-graded part being a Lie algebra. An F -fold symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the zero graded part in terms of the non-zero graded part.

Subsequently, a detailed analysis when $F = 3$ and for a specific extension of the Poincaré algebra was undertaken together with its explicit implementation in quantum field theory [20, 21, 22, 29]. However, this

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[§]Michel Goze and Adrian Tanasă have been partially supported by the Inter-University Cooperation Program of AUF (Agence Universitaire de la Francophonie).

general study reveled some difficulties that are not already resolved. This means that in order to understand the impact of these new structures, a general algebraic study should be undertaken. Thus, the aim of this paper is two-fold. Firstly a general classification of Lie algebra of order 3 is initiated when the zero graded part of the algebra is either (i) $\mathfrak{sl}(2)$ or (ii) the Poincaré algebra in four dimensions $\mathfrak{iso}(1, 3)$. It is then shown that the structure of the algebra is relatively rigid and a few examples of Lie algebras of order 3 are possible in the former case (see Theorem 3.1). Although in the latter cases, since the generators of the space-time translation commute, there are many possible extensions of the Poincaré algebra (see Theorem 4.8) Secondly, a theory of deformations and contractions is presented. This can be seen as a natural extension of the theory of contraction/deformation of Lie (super)algebras (see for example [12, 13]) to Lie algebras of order 3. Indeed, contraction/deformation are relevant in physics in the sense that they may provide a relationship between two different theories. For instance, the Poincaré algebra of special relativity and the Galilean algebra of non-relativistic physics are related through an Inönü-Wigner contraction. In the same way it is known that the N -extended supersymmetric extension of the Poincaré algebra can be obtained through a contraction of the superalgebra $\mathfrak{osp}(4|N)$. Similarly the extension of the Poincaré algebra studied in [20, 21, 22, 29] was obtained through a contraction of a certain Lie algebra of order 3 [26].

The content of the paper is organised as follow. In the next section the definition of Lie algebras of order F is recalled. Explicit examples are then given. In section three a classification of all Lie algebras of order 3 associated to $\mathfrak{sl}(2)$ is given. Section four is devoted to a general study of Lie algebras of order 3 associated to the Poincaré algebra in four dimensions $\mathfrak{iso}(1, 3)$. In section five the general notion (in a topological sense) of contractions is defined. This notion being too general more useful contractions (Inönü-Wigner contractions) are introduced. Section six is devoted to the implementation of the theory of deformations of Lie algebra of order 3 in the Gerstenhaber sense. Infinitesimal and isomorphic deformations are then introduced.

2 Lie algebras of order 3

In this section we recall the definition and some basic properties of Lie algebras of order F introduced in [26] and [27] and we define the algebraic variety of these algebraic structures.

2.1 Definition and examples of elementary Lie algebras of order 3

Definition 2.1 Let $F \in \mathbb{N}^*$. A \mathbb{Z}_F -graded \mathbb{C} -vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_{F-1}$ is called a complex Lie algebra of order F if

1. \mathfrak{g}_0 is a complex Lie algebra.
2. For all $i = 1, \dots, F-1$, \mathfrak{g}_i is a representation of \mathfrak{g}_0 . If $X \in \mathfrak{g}_0, Y \in \mathfrak{g}_i$ then $[X, Y]$ denotes the action of $X \in \mathfrak{g}_0$ on $Y \in \mathfrak{g}_i$ for all $i = 1, \dots, F-1$.
3. For all $i = 1, \dots, F-1$ there exists an F -linear, \mathfrak{g}_0 -equivariant map

$$\mu_i : S^F(\mathfrak{g}_i) \rightarrow \mathfrak{g}_0,$$

where $S^F(\mathfrak{g}_i)$ denotes the F -fold symmetric product of \mathfrak{g}_i , satisfying the following (Jacobi) identity

$$\sum_{j=1}^{F+1} [Y_j, \mu_i(Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_{F+1})] = 0,$$

for all $Y_j \in \mathfrak{g}_i, j = 1, \dots, F+1$.

Remark 2.2 If $F = 1$, by definition $\mathfrak{g} = \mathfrak{g}_0$ and a Lie algebra of order 1 is a Lie algebra. If $F = 2$, then \mathfrak{g} is a Lie superalgebra. Therefore, Lie algebras of order F appear as some kind of generalisations of Lie algebras and superalgebras.

Note that by definition the following Jacobi identities are satisfied by a Lie algebra of order F :

For any $X, X', X'' \in \mathfrak{g}_0$,

$$[[X, X'], X''] + [[X', X''], X] + [[X'', X], X'] = 0. \quad \text{J1}$$

For any $X, X' \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_i, i = 1, \dots, F-1$,

$$[[X, X'], Y] + [[X', Y], X] + [[Y, X], X'] = 0. \quad \text{J2}$$

For any $X \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_i, j = 1, \dots, F, i = 1, \dots, F-1$,

$$[X, \mu_i(Y_1, \dots, Y_F)] = \mu_i([X, Y_1], \dots, Y_F) + \dots + \mu_i(Y_1, \dots, [X, Y_F]). \quad \text{J3}$$

For any $Y_j \in \mathfrak{g}_i, j = 1, \dots, F+1, i = 1, \dots, F-1$,

$$\sum_{j=1}^{F+1} [Y_j, \mu_i(Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_{F+1})] = 0. \quad \text{J4}$$

Proposition 2.3 *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{F-1}$ be a Lie algebra of order F , with $F > 1$. For any $i = 1, \dots, F-1$, the \mathbb{Z}_F -graded vector spaces $\mathfrak{g}_0 \oplus \mathfrak{g}_i$ is a Lie algebra of order F . We call these type of algebras elementary Lie algebras of order F .*

In [26] an inductive process for the construction of Lie algebras of order F starting from a Lie algebra of order F_1 with $1 \leq F_1 < F$ is given. In this paper we are especially concerned by deformation and classification problems. Moreover, we restrict ourselves to elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We also denote the 3-linear map μ_1 by the 3-entries bracket $\{., ., .\}$ and we refer to it as a 3-bracket. Non-trivial examples of Lie algebras of order F (finite and infinite-dimensional) are given in [26] and [27]. We now give some examples of finite-dimensional Lie algebras of order 3, which will be relevant in the sequel.

Example 2.4 *Let $\mathfrak{g}_0 = \mathfrak{so}(2, 3)$ and \mathfrak{g}_1 its adjoint representation. Let $\{J_a, a = 1, \dots, 10\}$ be a basis of \mathfrak{g}_0 and $\{A_a = \text{ad}(J_a), a = 1, \dots, 10\}$ be the corresponding basis of \mathfrak{g}_1 . Thus, one has $[J_a, A_b] = \text{ad}([J_a, J_b])$. Let $g_{ab} = \text{Tr}(A_a A_b)$ be the Killing form. Then one can endow $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a Lie algebra of order 3 structure given by*

$$\{A_a, A_b, A_c\} = g_{ab}J_c + g_{ac}J_b + g_{bc}J_a.$$

Example 2.5 *Let \mathfrak{g}_0 be the Poincaré algebra in four dimensions and $\{L_{mn}, P_m : L_{mn} = -L_{nm}, m < n, m, n = 0, \dots, 3\}$ be a basis of \mathfrak{g}_0 with the non-zero brackets*

$$\begin{aligned} [L_{mn}, L_{pq}] &= \eta_{nq}L_{pm} - \eta_{mq}L_{pn} + \eta_{np}L_{mq} - \eta_{mp}L_{nq}, \\ [L_{mn}, P_p] &= \eta_{np}P_m - \eta_{mp}P_n. \end{aligned} \quad (2.1)$$

Let now \mathfrak{g}_1 be the 4-dimensional vector representation of \mathfrak{g}_0 , the action of \mathfrak{g}_0 on \mathfrak{g}_1 is given by

$$[L_{mn}, V_p] = \eta_{np}V_m - \eta_{mp}V_n, \quad [P_m, V_n] = 0$$

where $\{V_m : m = 0, \dots, 3\}$ is a basis of \mathfrak{g}_1 . The following brackets on \mathfrak{g}_1

$$\{V_m, V_n, V_r\} = \eta_{mn}P_r + \eta_{mr}P_n + \eta_{rn}P_m, \quad (2.2)$$

with the metric $\eta_{mn} = \text{diag}(1, -1, -1, -1)$ endow $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ with an elementary Lie algebra of order 3 structure.

2.2 The variety elementary $\mathcal{F}_{m,n}$ of Lie algebras of order 3

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an elementary Lie algebra of order 3 and let $A = (\mathfrak{g}_0 \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_0 \otimes \mathfrak{g}_1) \oplus \mathcal{S}^3(\mathfrak{g}_1)$. The multiplication of Lie algebra of order 3 is given by the linear map

$$\varphi : A \rightarrow \mathfrak{g}$$

satisfying the conditions J1-J4. Let $\varphi_1, \varphi_2, \varphi_3$ be the restrictions of φ to each of the terms of A

$$\varphi_1 : \mathfrak{g}_0 \otimes \mathfrak{g}_0 \rightarrow \mathfrak{g}_0,$$

$$\varphi_2 : \mathfrak{g}_0 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_1,$$

$$\varphi_3 : \mathcal{S}^3(\mathfrak{g}_1) \rightarrow \mathfrak{g}_0.$$

We denote this by $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.

Let $\{X_i : i = 1, \dots, m\}$ and resp. $\{Y_a : a = 1, \dots, n\}$ be a basis of \mathfrak{g}_0 and resp. \mathfrak{g}_1 . The maps φ_i ($i = 1, 2, 3$) are defined by their structure constants with regard to this basis

$$\varphi_1(X_i, X_j) = C_{ij}^k X_k, \quad \varphi_2(X_i, Y_b) = D_{ib}^c Y_c \text{ and } \varphi_3(Y_a, Y_b, Y_c) = E_{abc}^i X_i. \quad (2.3)$$

The structure constants $(C_{ij}^k, D_{ib}^c, E_{abc}^i)$ verify the following conditions:

$$\begin{aligned} C_{ij}^k &= -C_{ji}^k, \\ E_{abc}^i &= E_{acb}^i = E_{bac}^i = E_{bca}^i = E_{cab}^i = E_{cba}^i. \end{aligned} \quad (2.4)$$

and the polynomial equations corresponding to the Jacobi conditions J1-J4 are:

$$\begin{aligned} C_{ij}^\ell C_{\ell k}^m + C_{jk}^\ell C_{\ell i}^m + C_{ki}^\ell C_{\ell j}^m &= 0, \\ C_{ij}^k D_{ka}^c - D_{ja}^b D_{ib}^c + D_{ia}^b D_{jb}^c &= 0, \\ E_{abc}^j C_{ij}^k - D_{ia}^d E_{dbc}^k - D_{ib}^d E_{adc}^k - D_{ic}^d E_{abd}^k &= 0, \\ D_{ia}^\ell E_{bcd}^i + D_{ib}^\ell E_{cda}^i + D_{ic}^\ell E_{dab}^i + D_{id}^\ell E_{abc}^i &= 0. \end{aligned} \quad (2.5)$$

Let \mathbb{C}^N be the vector space whose elements are the N -tuple $(C_{ij}^k, E_{ij}^k, D_{ijk}^l)$, with $N = mC_m^2 + mn^2 + mC_n^3$. The polynomial equations (2.5) determine an algebraic variety $\mathcal{F}_{m,n}$ embedded in \mathbb{C}^N . Each point of $\mathcal{F}_{m,n}$ correspond to an $(m+n)$ -dimensional Lie algebra of order 3. Thus we identify any elementary Lie algebra of order 3 with the bracket φ to a point of $\mathcal{F}_{m,n}$.

Let us now consider the action of the group $GL(m, n) \cong GL(m) \times GL(n)$ on $\mathcal{F}_{m,n}$. For any $(h_0, h_1) \in GL(m, n)$, this action is defined by

$$(h_0, h_1) \cdot (\varphi_1, \varphi_2, \varphi_3) \rightarrow (\varphi'_1, \varphi'_2, \varphi'_3)$$

where

$$\begin{aligned} \varphi'_1(X_1, X_2) &= h_0^{-1} \varphi_1(h_0(X_1), h_0(X_2)), \\ \varphi'_2(X_1, Y_2) &= h_1^{-1} \varphi_2(h_0(X_1), h_1(Y_2)), \\ \varphi'_3(Y_1, Y_2, Y_3) &= h_0^{-1} \varphi_3(h_1(Y_1), h_1(Y_2), h_1(Y_3)), \end{aligned} \quad (2.6)$$

where (Y_1, Y_2, Y_3) represents an element of $\mathcal{S}^3(\mathfrak{g}_1)$. The group $GL(m, n)$ can be embedded in $GL(n+m)$. It corresponds to the subgroup of $GL(m+n)$ which let invariant the subspaces \mathfrak{g}_0 and \mathfrak{g}_1 of \mathfrak{g} . Denote by \mathcal{O}_φ the orbit of $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with respect to this action. Then the algebraic variety $\mathcal{F}_{m,n}$ is fibered by theses orbits. The quotient set is the set of isomorphism classes of $(m+n)$ -dimensional elementary Lie algebras of order 3.

3 $\mathfrak{sl}(2)$ -algebras of order 3

In this section we study complex Lie algebras of order 3, $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ for which $\mathfrak{g}_0 \cong \mathfrak{sl}(2)$ and \mathfrak{g}_1 is an arbitrary representation of $\mathfrak{sl}(2)$. We denote by X_+, X_-, X_0 a standard basis of \mathfrak{g}_0

$$[X_0, X_+] = 2X_+, \quad [X_0, X_-] = -2X_-, \quad [X_+, X_-] = X_0, \quad (3.1)$$

and \mathcal{D}_ℓ ($\ell \in \mathbb{N}$) an irreducible representation of dimension $\ell + 1$.

Theorem 3.1 *The graded complex vector space $\mathfrak{g} \cong \mathfrak{sl}(2) \oplus \mathfrak{g}_1$, with \mathfrak{g}_1 a representation of $\mathfrak{sl}(2)$ is provided with a non-trivial Lie algebra structure of order 3 if and only if:*

1. $\mathfrak{g}_1 \cong \mathcal{D}_2$ ($\mathcal{D}_2 = \langle Y_2, Y_0, Y_{-2} \rangle$), with the non-zero three-brackets

$$\begin{aligned} \{Y_2, Y_{-2}, Y_0\} &= X_0, & \{Y_0, Y_0, Y_0\} &= 6X_0, \\ \{Y_2, Y_{-2}, Y_2\} &= 2X_+, & \{Y_2, Y_0, Y_0\} &= 2X_+, \\ \{Y_{-2}, Y_2, Y_{-2}\} &= 2X_-, & \{Y_{-2}, Y_0, Y_0\} &= 2X_-. \end{aligned} \quad (3.2)$$

2. $\mathfrak{g}_1 \cong \mathcal{D}_2 \oplus \mathcal{D}_0^{(1)} \oplus \dots \oplus \mathcal{D}_0^{(k)}$ ($\mathcal{D}_2 = \langle Y_2, Y_0, Y_{-2} \rangle$, $\mathcal{D}_0^{(k)} = \langle \lambda_k \rangle$), with the non-zero three-brackets

$$\begin{aligned} \{\lambda_i, \lambda_j, Y_2\} &= \alpha_{ij} X_+, \\ \{\lambda_i, \lambda_j, Y_0\} &= \alpha_{ij} X_0, \\ \{\lambda_i, \lambda_j, Y_{-2}\} &= \alpha_{ij} X_-. \end{aligned} \quad (3.3)$$

and $\alpha_{ij} \in \mathbb{C}$.

3. $\mathfrak{g}_1 \cong \mathcal{D}_1 \oplus \mathcal{D}_0$ ($\mathcal{D}_1 = \langle Y_1, Y_{-1} \rangle$, $\mathcal{D}_0 = \langle \lambda \rangle$), with the non-zero three-brackets

$$\begin{aligned} \{\lambda, Y_1, Y_1\} &= -2X_+, \\ \{\lambda, Y_1, Y_{-1}\} &= X_0, \\ \{\lambda, Y_{-1}, Y_{-1}\} &= 2X_-. \end{aligned} \quad (3.4)$$

Proof. Since $\mathfrak{g}_1 = \bigoplus_k \mathcal{D}_{\ell_k}$, with \mathcal{D}_{ℓ_k} an irreducible representation of dimension $\ell_k + 1$, the 3-brackets $\{\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1\}$ contain terms like (i) $\{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_1}\}$, (ii) $\{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}\}$, (iii) $\{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}, \mathcal{D}_{\ell_3}\}$.

I. Consider firstly the case $\{\mathcal{D}_\ell, \mathcal{D}_\ell, \mathcal{D}_\ell\}$. A simple weight argument shows that ℓ is even, furthermore the non-vanishing three brackets are

$$\{Y_i, Y_j, Y_k\} = \begin{cases} \alpha_{ijk} X_+ & i+j+k=2, \\ \beta_{ijk} X_0 & i+j+k=0, \\ \gamma_{ijk} X_- & i+j+k=-2. \end{cases}$$

Suppose firstly that $\ell = 2$. The action of $\mathfrak{sl}(2)$ on \mathcal{D}_2 is

$$\begin{aligned} [X_0, Y_{-1}] &= -2Y_{-1}, & [X_-, Y_0] &= 2Y_{-1}, & [X_-, Y_1] &= -Y_0, \\ [X_+, Y_{-1}] &= Y_0, & [X_+, Y_0] &= -2Y_1, & [X_0, Y_1] &= 2Y_1. \end{aligned} \quad (3.5)$$

From symmetry considerations one has $\alpha_{ijk} = \gamma_{-i, -j, -k}$, and the Jacobi identity J3 gives $\alpha_{1,1,-1} = \alpha_{1,0,0} = \gamma_{-1,-1,1} = \gamma_{-1,0,0} = 2t$, $\beta_{1,-1,0} = t$, $\beta_{0,0,0} = 6t$, with $t \in \mathbb{C}$. Furthermore, a direct calculus shows that the Jacobi identities J4 are satisfied for any t . If $t = 0$, the Lie algebra of order 3 is trivial. If $t \neq 0$ all the algebras are equivalent. Since for $\mathfrak{sl}(2)$ the Casimir operator is given by $Q = \frac{1}{2}H^2 + X_+X_- + X_-X_+$ we have $\text{Tr}(X_+X_-) = g_{+-} = g_{-+} = 1$, $\text{Tr}(X_0X_0) = g_{00} = 2$ and the three-brackets (3.2) can be rewritten [26]

$$\{Y_i, Y_j, Y_k\} = g_{ij}X_k + g_{jk}X_i + g_{ki}X_j,$$

(here X_+, X_0, X_- are denoted X_2, X_0, X_{-2}).

Now we assume that $\ell > 2$ and we prove that $\{\mathcal{D}_\ell, \mathcal{D}_\ell, \mathcal{D}_\ell\} = 0$.

1. The bracket $\{Y_0, Y_0, Y_0\} = 0$ as a consequence of J4 applied to (Y_ℓ, Y_0, Y_0, Y_0) .
2. The bracket $\{Y_0, Y_0, Y_i\} = 0$ as a consequence of J4 applied to (Y_0, Y_0, Y_0, Y_i) .
3. The bracket $\{Y_0, Y_i, Y_j\} = 0$, $i + j \neq 0$ as a consequence of J4 applied to (Y_0, Y_0, Y_i, Y_j) .
4. The bracket $\{Y_0, Y_i, Y_{-i}\} = 0$. If $i \neq 2$ this is a consequence of J4 applied to (Y_0, Y_i, Y_i, Y_{-i}) and if $i = 2$ this is a consequence of J4 applied to $(Y_\ell, Y_0, Y_2, Y_{-2})$.
5. The bracket $\{Y_i, Y_j, Y_k\} = 0$, $i + j + k \neq 0$ as a consequence of J4 applied to (Y_0, Y_i, Y_j, Y_k) .
6. The bracket $\{Y_i, Y_j, Y_{-i-j}\} = 0$ as a consequence of J4 applied to $(Y_k, Y_i, Y_j, Y_{-i-j})$ with $k \neq i, j, -i - j, 0$. Such a k always exists since the dimension of \mathcal{D}_ℓ is bigger than 5.

II. Consider now the case $\{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}\}$.

- We prove now that $\{\mathcal{D}_\ell, \mathcal{D}_\ell, \mathcal{D}_0\} = 0$ for all ℓ but $\ell = 1$.

1. It is easy to see that $\ell = 0$ leads to a trivial Lie algebra of order 3.
2. If $\ell = 1$ we denote $\mathcal{D}_0 = \langle \lambda \rangle$ and $\mathcal{D}_1 = \langle Y_1, Y_{-1} \rangle$. The action of $\mathfrak{sl}(2)$ on \mathcal{D}_1 is

$$\begin{aligned} [X_0, Y_1] &= Y_1, & [X_+, Y_{-1}] &= Y_1, \\ [X_-, Y_1] &= Y_{-1}, & [X_0, Y_{-1}] &= -Y_{-1}, \end{aligned}$$

and a simple weight argument gives

$$\{\lambda, Y_1, Y_1\} = \alpha X_+, \quad \{\lambda, Y_1, Y_{-1}\} = \beta X_0, \quad \{\lambda, Y_{-1}, Y_{-1}\} = \gamma X_-.$$

The Jacobi identities J3 and J4 give $\alpha = -2t, \beta = t, \gamma = 2t$, $t \in \mathbb{C}$.

3. If $\ell = 2$ using the Clebsch-Gordan decomposition $\mathcal{D}_2 \otimes \mathcal{D}_2 = \mathcal{D}_4 \oplus \mathcal{D}'_2 \oplus \mathcal{D}_0$, since the representation \mathcal{D}'_2 is antisymmetric in the permutation of the two factors \mathcal{D}_2 , we have $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}_0\} = 0$.
4. If $\ell > 2$, we denote $\mathcal{D}_\ell = \langle Y_\ell, Y_{\ell-2}, \dots, Y_{-\ell} \rangle$. A simple weight arguments shows that the possible non-vanishing 3-brackets are: $\{\lambda, Y_i, Y_{-i+2}\}$, $\{\lambda, Y_i, Y_{-i}\}$ and $\{\lambda, Y_i, Y_{-i-2}\}$.

- (a) The brackets $\{\lambda, Y_i, Y_{-i+2}\} = 0$, $i \neq 0, -2, \ell$ as a consequence of J4 applied to $(\lambda, Y_i, Y_i, Y_{-i+2})$.
- (b) The bracket $\{\lambda, Y_0, Y_2\} = 0$ as a consequence of J4 applied to (λ, Y_0, Y_2, Y_2) .
- (c) The brackets $\{\lambda, Y_{-2}, Y_4\} = 0$ as a consequence of J4 applied to $(\lambda, Y_{-2}, Y_{-2}, Y_4)$.
- (d) The brackets $\{\lambda, Y_\ell, Y_{-\ell+2}\} = 0$ as a consequence of J4 applied to $(\lambda, Y_\ell, Y_{-\ell+2}, Y_{-\ell+2})$.
- (e) The brackets $\{\lambda, Y_i, Y_{-i}\} = 0$, $i \neq 0$ as a consequence of J4 applied to $(\lambda, Y_i, Y_i, Y_{-i})$.
- (f) The bracket $\{\lambda, Y_0, Y_0\} = 0$, as a consequence of J4 applied to $(\lambda, Y_0, Y_0, Y_\ell)$.
- (g) The brackets $\{\lambda, Y_i, Y_{-i-2}\}$ go along the same line as the brackets $\{\lambda, Y_i, Y_{-i+2}\}$.

- We now consider the brackets of type $\{\mathcal{D}_\ell, \mathcal{D}_0, \mathcal{D}_0\}$. By a weight argument and identity J4 one has $\ell = 2$ and the non-trivial 3-brackets are:

$$\begin{aligned} \{\lambda, \lambda, Y_2\} &= \alpha X_+, \\ \{\lambda, \lambda, Y_0\} &= \alpha X_0, \\ \{\lambda, \lambda, Y_{-2}\} &= \alpha X_-. \end{aligned}$$

- We now prove that $\{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}\} = 0$ with $\ell_1, \ell_2 \neq 0$.

1. Let $\ell_1 = \ell_2 = 2$, we denote $\mathcal{D}_2 = \langle Y_2, Y_0, Y_{-2} \rangle$, $\mathcal{D}'_2 = \langle Y'_2, Y'_0, Y'_{-2} \rangle$ the two three-dimensional representations. In this case, we have four types of brackets: $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}_2\}$, $\{\mathcal{D}'_2, \mathcal{D}'_2, \mathcal{D}'_2\}$, $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}'_2\}$, $\{\mathcal{D}_2, \mathcal{D}'_2, \mathcal{D}'_2\}$. The possible non-vanishing three-brackets are

$$\begin{aligned} \{Y_i, Y_j, Y_k\} &= \alpha g_{ij} X_k + \alpha g_{jk} X_i + \alpha g_{ki} X_j, \\ \{Y'_i, Y'_j, Y'_k\} &= \alpha' g_{ij} X_k + \alpha' g_{jk} X_i + \alpha' g_{ki} X_j, \\ \{Y_2, Y_2, Y'_{-2}\} &= \alpha_1 X_+ \quad \{Y'_2, Y_2, Y_{-2}\} = \alpha_2 X_+ \quad \{Y_2, Y_0, Y'_0\} = \alpha_3 X_+ \quad \{Y'_2, Y_0, Y_0\} = \alpha_4 X_+ \\ \{Y_{-2}, Y_{-2}, Y'_2\} &= \alpha_1 X_- \quad \{Y'_{-2}, Y_{-2}, Y_2\} = \alpha_2 X_- \quad \{Y_{-2}, Y_0, Y'_0\} = \alpha_3 X_- \quad \{Y'_{-2}, Y_0, Y_0\} = \alpha_4 X_- \\ \{Y_2, Y_{-2}, Y'_0\} &= \beta_1 X_0 \quad \{Y_2, Y_0, Y'_{-2}\} = \beta_2 X_0 \quad \{Y'_2, Y_{-2}, Y_0\} = \beta_3 X_0 \quad \{Y_0, Y_0, Y'_0\} = \beta_4 X_0 \end{aligned} \quad (3.6)$$

(plus similar terms with $\{Y', Y', Y\}$ and coefficients $\alpha'_1, \dots, \beta'_4$, as $\{Y'_2, Y'_2, Y_{-2}\} = \alpha'_1 X_+$, etc.). Working out the Jacobi identity J4, one gets that the coefficients $\alpha, \alpha_1, \dots, \beta_4, \alpha', \alpha'_1, \dots, \beta'_4$ must be zero. In fact one has

- (a) $(Y_2, Y_2, Y'_{-2}, Y'_{-2})$ gives $\alpha_1 = \alpha'_1 = 0$;
- (b) $(Y_{-2}, Y_{-2}, Y_2, Y'_2)$ gives $\alpha = \alpha_2 = 0$ (resp. $\alpha' = \alpha'_2 = 0$);
- (c) (Y'_0, Y'_0, Y_0, Y_2) gives $\alpha_3 = 0$ (resp. $\alpha'_3 = 0$);
- (d) (Y_0, Y_0, Y_0, Y'_2) gives $\alpha_4 = 0$ (resp. $\alpha'_4 = 0$);
- (e) (Y_2, Y_2, Y_{-2}, Y'_0) gives $\beta_1 = 0$ (resp. $\beta'_1 = 0$);
- (f) (Y_0, Y_0, Y_2, Y'_{-2}) gives $\beta_2 = 0$ (resp. $\beta'_2 = 0$);
- (g) $(Y_{-2}, Y_0, Y'_2, Y'_2)$ gives $\beta_3 = 0$ (resp. $\beta'_3 = 0$);
- (h) (Y_2, Y_0, Y_0, Y'_0) gives $\beta_4 = 0$ (resp. $\beta'_4 = 0$).

Thus all the brackets vanishes.

2. $\ell_1 \neq 2, \ell_2$ arbitrary $\mathcal{D}_{\ell_1} = \langle Y_{\ell_1}, Y_{\ell_1-2}, \dots, Y_{-\ell_1} \rangle$ and $\mathcal{D}_{\ell_2} = \langle Y'_{\ell_2}, Y'_{\ell_2-2}, \dots, Y'_{-\ell_2} \rangle$. We consider the bracket $\{Y_i, Y_j, Y'_k\}$. The identity J4 applied to (Y_i, Y_i, Y_j, Y'_k) leads to the vanishing of the 3-brackets except when $2i + k \neq 0, \pm 2$ or $(i \neq \ell$ and $i + j + k \neq 2)$. The identity J4 applied to (Y_i, Y_j, Y_j, Y'_k) leads to the vanishing of the 3-brackets except when $2j + k \neq 0, \pm 2$ or $(j \neq \ell$ and $i + j + k \neq 2)$. The cases that remains to be studied are:

- (a) If $k = -2j + 2 = -2i + 2$ then $i = j$ and the bracket $\{Y_i, Y_i, Y'_{-2i+2}\}$ vanishes.
- (b) If $k = -2j + 2 = -2i$ then $i = j - 1$ which is not possible.
- (c) If $k = -2j + 2 = -2i - 2$ then $j = i + 2$ and the bracket reduces to $\{Y_i, Y_{i+2}, Y'_{-2i-2}\}$. Then identity J4 applied to $(Y_{\ell_1}, Y_i, Y_{i+2}, Y'_{-2i-2})$ and to $(Y_{-\ell_1}, Y_i, Y_{i+2}, Y'_{-2i-2})$ leads to $\{Y_i, Y_{i+2}, Y'_{-2i-2}\} = 0$.
- (d) If $k = -2j = -2i$ then $i = j$ and the bracket vanishes as before. If $k = -2j = -2i - 2$ then $j = i + 1$ which is also excluded. If $k = -2j - 2 = -2i - 2$ then $i = j$ and the bracket vanishes as before.

3. Let $\ell_1 = 2, \ell_2 \neq 2$ and consider the brackets of the type $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}_{\ell_2}\}$. The Jacobi identity J4 applied on $(Y_{i_1}, Y_{i_2}, Y'_{i_3}, Y'_{i_3})$ with $Y_{i_1}, Y_{i_2} \in \mathcal{D}_2$ and $Y_{i_3} \in \mathcal{D}_3$ leads to $\{\mathcal{D}_2, \mathcal{D}_2, \mathcal{D}_{\ell_2}\} = 0$.

III. Consider now the case $\{\mathcal{D}_{\ell_1}, \mathcal{D}_{\ell_2}, \mathcal{D}_{\ell_3}\}$.

If $\ell_1 = \ell_2 = 0$, by weight arguments we have $\ell_3 = 2$ and the possible non-vanishing three brackets are

$$\begin{aligned} \{\lambda_1, \lambda_2, Y_2\} &= \alpha_{12} X_+, \\ \{\lambda_1, \lambda_2, Y_0\} &= \alpha_{12} X_0, \\ \{\lambda_1, \lambda_2, Y_{-2}\} &= \alpha_{12} X_-. \end{aligned}$$

where $\mathcal{D}_0^{(1)} = \langle \lambda_1 \rangle$, $\mathcal{D}_0^{(2)} = \langle \lambda_2 \rangle$ and $\mathcal{D}_2 = \langle Y_2, Y_0, Y_{-2} \rangle$.

- If $\ell_1 = 0$ and $\ell_2, \ell_3 \neq 0$ the Jacobi identity J4 applied to (λ, Y, Y') with $\lambda \in \mathcal{D}_0, Y \in \mathcal{D}_{\ell_2}, Y' \in \mathcal{D}_{\ell_3}$ leads to $\{\lambda, Y, Y'\} = 0$ except if $\ell_2 = \ell_3 = 1$.
- If $\ell_1 = 0$ and $\ell_2 = \ell_3 = 1$, denoting $\mathcal{D}_1 = \langle Y_1, Y_{-1} \rangle, \mathcal{D}'_1 = \langle Y'_1, Y'_{-1} \rangle, \mathcal{D}_0 = \langle \lambda \rangle$, the non-vanishing brackets are

$$\begin{aligned} \{Y_1, Y_1, \lambda\} &= -2\alpha X_+, \{Y_1, Y_{-1}, \lambda\} = \alpha X_0, \{Y_{-1}, Y_{-1}, \lambda\} = 2\alpha X_- \\ \{Y'_1, Y'_1, \lambda\} &= -2\alpha' X_+, \{Y'_1, Y'_{-1}, \lambda\} = \alpha' X_0, \{Y'_{-1}, Y'_{-1}, \lambda\} = 2\alpha' X_- \\ \{Y_1, Y'_1, \lambda\} &= \beta_1 X_+, \{Y_1, Y'_{-1}, \lambda\} = \beta_2 X_0, \{Y'_1, Y_{-1}, \lambda\} = \beta_3 X_0, \{Y_{-1}, Y'_{-1}, \lambda\} = \beta_4 X_- \end{aligned}$$

The Jacobi identity J4 with $(\lambda, Y_1, Y_1, Y'_{-1})$ implies $\alpha = \beta_2 = 0$, with $(\lambda, Y'_1, Y'_1, Y_{-1})$ implies $\alpha' = \beta'_2 = 0$, with $(\lambda, Y_1, Y_{-1}, Y'_1)$ implies $\beta_1 = 0$ and with $(\lambda, Y_{-1}, Y_1, Y'_{-1})$ implies $\beta_4 = 0$.

- If $\ell_1, \ell_2, \ell_3 \neq 0$, let $Y \in \mathcal{D}_{\ell_1}, Y' \in \mathcal{D}_{\ell_2}, Y'' \in \mathcal{D}_{\ell_3}$. Then J4 applied to (Y, Y, Y', Y'') leads to $\{Y, Y', Y''\} = 0$.

Taking all the cases obtained above, the only Lie algebras of order 3 associated to $\mathfrak{sl}(2)$ are $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathcal{D}_2$, $\mathfrak{g} = \mathfrak{sl}(2) \oplus D_2 \oplus D_0^{(1)} \oplus \dots \oplus D_0^{(k)}$ or $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathcal{D}_0 \oplus \mathcal{D}_1$ with brackets given in (3.2), (3.3) and (3.4). QED.

Remark 3.2 In [26] two families of algebras associated to $\mathfrak{sp}(n)$ were constructed. We have however checked that they coincide when $n = 1$ i.e. for $\mathfrak{sl}(2)$. This algebra is the one given in Eq.[3.2]. The algebra (3.3) was also obtained in [26] and the algebra (3.4) in [26] and [1].

4 Poincaré algebras of order 3

In this section we study and provide a systematic method to obtain all elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where \mathfrak{g}_0 is isomorphic to the Poincaré algebra and \mathfrak{g}_1 is an arbitrary finite dimensional representation of the Poincaré algebra. Such algebras are called *Poincaré algebras of order 3*.

To construct these algebras, we proceed in several steps. Firstly, we extend the action of $\mathfrak{so}(1, 3, \mathbb{C}) = \mathfrak{so}(1, 3) \otimes_{\mathbb{R}} \mathbb{C}$ on \mathfrak{g}_1 , with \mathfrak{g}_1 a finite dimensional representation of $\mathfrak{so}(1, 3, \mathbb{C})$, to the action of the complexified Poincaré algebra on \mathfrak{g}_1 . Then, we construct the $\mathfrak{so}(1, 3, \mathbb{C})$ -equivariant mappings from $\mathcal{S}^3(\mathfrak{g}_1)$ into $\mathcal{D}_{1,1}$, with $\mathcal{D}_{1,1}$ the vector representation of $\mathfrak{so}(1, 3, \mathbb{C})$. Finally, we obtain all Lie algebras of order 3, $\mathfrak{g} = (\mathfrak{so}(1, 3, \mathbb{C}) \oplus \mathcal{D}_{1,1}) \oplus \mathfrak{g}_1$.

4.1 Finite dimensional representations of the Poincaré algebra

The Poincaré algebra in $(1+3)$ dimensions $\mathfrak{iso}(1, 3)$ (see Example 2.5 for notations) is given by

$$\begin{aligned} [L_{mn}, L_{pq}] &= \eta_{nq} L_{pm} - \eta_{mq} L_{pn} + \eta_{np} L_{mq} - \eta_{mp} L_{nq}, \\ [L_{mn}, P_p] &= \eta_{np} P_m - \eta_{mp} P_n, \quad [P_m, P_n] = 0, \end{aligned} \tag{4.1}$$

where η_{mn} is the Minkowski metric. Let $\mathfrak{iso}(1, 3, \mathbb{C}) = \mathfrak{iso}(1, 3) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified of $\mathfrak{iso}(1, 3)$. Its Levi part is isomorphic to $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Consider in (4.1) the following change of basis

$$\begin{aligned} U_0 &= iL_{12} - L_{03}; & V_0 &= iL_{12} + L_{03}; \\ U_+ &= \frac{1}{2}(iL_{23} - L_{31} - L_{01} - iL_{02}); & V_+ &= \frac{1}{2}(iL_{23} - L_{31} + L_{01} + iL_{02}); \\ U_- &= \frac{1}{2}(iL_{23} + L_{31} - L_{01} + iL_{02}); & V_- &= \frac{1}{2}(iL_{23} + L_{31} + L_{01} - iL_{02}); \\ \begin{pmatrix} p_{+-} & p_{--} \\ p_{++} & p_{-+} \end{pmatrix} &= P_m \sigma^m = \begin{pmatrix} P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_3 & P_0 - P_3 \end{pmatrix}, \end{aligned} \tag{4.2}$$

(with σ^0 the identity matrix and σ^i , $i = 1, 2, 3$ the Pauli matrices). In this basis the $\mathfrak{iso}(1, 3, \mathbb{C})$ brackets are given by

$$\begin{aligned}
[U_0, U_\pm] &= \pm 2U_\pm, & [V_0, V_\pm] &= \pm 2V_\pm, \\
[U_+, U_-] &= U_0, & [V_+, V_-] &= V_0, \\
[U_+, p_{-\varepsilon}] &= p_{+\varepsilon}, & [V_+, p_{\varepsilon-}] &= -p_{\varepsilon+}, \\
[U_-, p_{+\varepsilon}] &= p_{-\varepsilon}, & [V_-, p_{\varepsilon+}] &= -p_{\varepsilon-}, \\
[U_0, p_{\varepsilon\varepsilon'}] &= \varepsilon p_{\varepsilon\varepsilon'}, & [V_0, p_{\varepsilon\varepsilon'}] &= \varepsilon' p_{\varepsilon\varepsilon'},
\end{aligned} \tag{4.3}$$

(with $\varepsilon, \varepsilon' = \pm$).

Let \mathcal{D}_i be the irreducible $(i+1)$ -dimensional representation of $\mathfrak{sl}(2)$. We note by $\mathcal{D}_{i,j} = \mathcal{D}_i \otimes \mathcal{D}_j$, $i, j \in \mathbb{N}$ the irreducible representation (since there is a factor 2 in the first line on equation (4.3) i belongs to \mathbb{N} and not to $\frac{1}{2}\mathbb{N}$) of dimension $(i+1)(j+1)$ of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ defined from

$$\rho(U_0 + V_0)(x \otimes y) = [U_0, x] \otimes y + x \otimes [V_0, y].$$

Let $\mathfrak{g}_1 = \oplus_k \mathcal{D}_{i_k, j_k}$ be an arbitrary reducible representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

Lemma 4.1 *Let \mathcal{D} be a (finite dimensional) representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. The action of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ on \mathcal{D} extends to an action of $\mathfrak{iso}(1, 3, \mathbb{C})$ on \mathcal{D} such that:*

1. The operators $\rho(P_m)$ ($m = 0, \dots, 3$), are nilpotent.
2. Let

$$A_p = \bigcap_{p_0 + p_1 + p_2 + p_3 = p} \text{Ker} \left((\rho(P_0))^{p_0} (\rho(P_1))^{p_1} (\rho(P_2))^{p_2} (\rho(P_3))^{p_3} \right),$$

then, there exists an N such that we have the filtration

$$A_1 \subset A_2 \subset \dots \subset A_N = \mathcal{D},$$

and for every $0 \leq p \leq N$, A_p is an $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ module.

3. \mathcal{D} is indecomposable (i.e. one can find an irreducible representation $\mathcal{D}' \subset \mathcal{D}$ of $\mathfrak{iso}(1, 3)$ such that it is impossible to have $\mathcal{D} = \mathcal{D}' \oplus \mathcal{D}''$ where \mathcal{D}'' is stable under the action $\rho(P_m)$ –see Examples 4.2 below –),

$$0 \xleftarrow{\rho(P_m)} A_1 \xleftarrow{\rho(P_m)} A_2 \xleftarrow{\rho(P_m)} \dots \xleftarrow{\rho(P_m)} A_N = \mathcal{D}.$$

Let $B_p^m = \rho(P_m)(A_p)$. Then $B_p^0 = B_p^i$ for $i = 1, 2, 3$. We denote by B_p this space and we have

$$B_p \subset A_p \otimes \mathcal{D}_{1,1} \subset A_{p-1}.$$

Proof. 1. Let λ_0 be an arbitrary eigenvalue of $\rho(P_0)$ and $E_0 = \text{Ker}(\lambda_0 - \rho(P_0)) \subseteq \text{Ker}(\lambda_0 - \rho(P_0))^{n_0} \subseteq \mathcal{D}$ (with $\text{Ker}(\lambda_0 - \rho(P_0))^{n_0}$ the generalised eigenspace). Denote λ_i ($i = 1, 2, 3$) an arbitrary eigenvalue of $\rho(P_i)_{|_{E_0}}$ and let $V = \text{Ker}(\lambda_i - \rho(P_i)_{|_{E_0}}) \subseteq \left(\lambda_i - \rho(P_i)_{|_{E_0}} \right)^{n_i} \subseteq E_0$. Since $[\rho(P_i), \rho(P_0)] = 0$ then E_0 is invariant by the action of P_i . In V we have $\rho(P_0)_{|_V} = \lambda_0 \text{Id}$, $\rho(P_i)_{|_V} = \lambda_i \text{Id}$, since $[L_{i0}, P_i] = P_0$ we have $\lambda_0 = \lambda_i = 0$. And $\rho(P_0), \rho(P_i)$ are nilpotent operators.

2. Since the operators $\rho(P_m)$ are nilpotent (we denote n_m the index of nilpotency of $\rho(P_m)$) and are commuting operators, it is obvious that there exists an $N \geq \max(n_0, n_1, n_2, n_3)$ such that $A_N = \mathcal{D}$. Furthermore, since for all $L \in \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, there exists a $P \in \mathcal{D}_{1,1}$ such that $[L, P] = P_0$ we have $[L, P^p] = pP_0 P^{p-1}$. This means that if $v \in A_p$, $[L, v] \in A_p$. Thus, A_p is an $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ module.

3. Let $w \in B_p^0$, then there exists $v \in A_p$ such that $w = [P_0, v]$. Since A_p is an $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ module we have $[L_{0i}, v'] = v$ ($i = 1, 2, 3$) with $v' \in A_p$. Using the Jacobi identity of Lie algebras, $w = [P_0, v] = [P_0, [L_{0i}, v']]$ leads to $[P_i, v'] = w - [L_{0i}, [P_0, v']]$ and $B_p^i \subseteq B_p^0$. The converse goes along the same lines and we have $B_p^0 = B_p^i$ for $i = 1, 2, 3$. Finally, the $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ equivariance of the mapping guarantees that, $B_p \subset A_p \otimes \mathcal{D}_{1,1} \subset A_{p-1}$ and \mathcal{D} is indecomposable. QED.

Example 4.2 (1) If $\mathfrak{g}_1 = \mathcal{D}_{1,1} \oplus \mathcal{D}_{0,0}$ the vector plus the scalar representations of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, P_m can be represented by 5×5 nilpotent matrices. If we denote $\langle v_m, m = 0, \dots, 3 \rangle$ (resp. $\langle w_0 \rangle$) a basis of $\mathcal{D}_{1,1}$ (resp. $\mathcal{D}_{0,0}$) we can define

$$\rho(P_m)w_0 = v_m, \quad \rho(P_m)v_n = 0.$$

(2) If $\mathfrak{g}_1 = \mathcal{D}_{1,0} \oplus \mathcal{D}_{0,1}$, since $\mathcal{D}_{1,0} \otimes \mathcal{D}_{1,1} = \mathcal{D}_{2,1} \oplus \mathcal{D}_{0,1} \supset \mathcal{D}_{0,1}$, P_m can be represented by the 4×4 matrices $\rho(P_m) = \begin{pmatrix} 0 & \sigma_m \\ 0 & 0 \end{pmatrix}$ such that for $\psi \in \mathcal{D}_{1,0}, \bar{\chi} \in \mathcal{D}_{0,1}$ we have

$$\rho(P_m)\psi = 0, \quad \rho(P_m)\bar{\chi} = \sigma_m\psi.$$

(3) The example above can be even refined. Let $\mathfrak{g}_1 = \mathcal{D}_{1,0} \oplus \mathcal{D}_{0,1} \oplus \mathcal{D}'_{0,1}$. The action of the P 's can be defined as follow:

$$\rho(P_m)\psi = 0, \rho(P_m)\psi' = 0 \quad \rho(P_m)\bar{\chi} = \sigma_m\psi,$$

with $\psi \in \mathcal{D}_{1,0}, \psi' \in \mathcal{D}'_{1,0}, \bar{\chi} \in \mathcal{D}_{0,1}$. Here, $\text{Ker}(\text{ad } P_m) = \mathcal{D}_{1,0} \oplus \mathcal{D}'_{1,0}$, $\text{Ker}(\text{ad } P_m)^2 = \mathcal{D}_{1,0} \oplus \mathcal{D}_{0,1} \oplus \mathcal{D}'_{0,1}$ ($m = 0, \dots, 3$) and $B_2 = \mathcal{D}_{1,0}$.

Remark 4.3 If \mathfrak{g}_1 is an irreducible representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, then the action of $\text{ad } P$ on \mathfrak{g}_1 is trivial. Indeed, since \mathfrak{g}_1 is irreducible, $\text{Ker}(\text{ad } P_m)$ is equal either to \mathfrak{g}_1 or $\{0\}$. But since $\text{ad } P_0, \dots, \text{ad } P_3$ commute they can be simultaneously diagonalised this means that $\text{Ker}(\text{ad } P_m) \neq \{0\}$ and the action of $\text{ad } P$ on \mathfrak{g}_1 is trivial.

4.2 $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings

Now, we construct the possible $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings from $\mathcal{S}^3(\mathfrak{g}_1)$ into $\mathcal{D}_{1,1}$, with \mathfrak{g}_1 an arbitrary representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. We recall the following isomorphisms of representations of $GL(A) \times GL(B)$ [8]:

$$\begin{aligned} \mathcal{S}^p(A \oplus B) &= \bigoplus_{k=0}^p \mathcal{S}^k(A) \otimes \mathcal{S}^{p-k}(B) \\ \mathcal{S}^p(A \otimes B) &= \bigoplus_{\Gamma} \mathcal{S}^{\Gamma}(A) \otimes \mathcal{S}^{\Gamma}(B), \end{aligned} \quad (4.4)$$

where the second sum is taken over all Young diagrams Γ of length p and $\mathcal{S}^{\Gamma}(A)$ denotes the irreducible representation of $GL(A)$ corresponding to the Young symmetriser of Γ . In particular this gives

$$\begin{aligned} \mathcal{S}^3(A \oplus B \oplus C) &= \mathcal{S}^3(A) \oplus \mathcal{S}^3(B) \oplus \mathcal{S}^3(C) \oplus A \otimes B \otimes C \oplus \\ \mathcal{S}^2(A) \otimes B \oplus \mathcal{S}^2(A) \otimes C \oplus \mathcal{S}^2(B) \otimes A \oplus \mathcal{S}^2(B) \otimes C \oplus \mathcal{S}^2(C) \otimes A \oplus \mathcal{S}^2(C) \otimes B & \quad (4.5) \\ \mathcal{S}^2(A \otimes B) &= \mathcal{S}^2(A) \otimes \mathcal{S}^2(B) \oplus \Lambda^2(A) \otimes \Lambda^2(B) \\ \mathcal{S}^3(A \otimes B) &= \mathcal{S}^3(A) \otimes \mathcal{S}^3(B) \oplus \mathcal{S}^{\begin{smallmatrix} \square & \square \end{smallmatrix}}(A) \otimes \mathcal{S}^{\begin{smallmatrix} \square & \square \end{smallmatrix}}(B) \oplus \Lambda^3(A) \otimes \Lambda^3(B). \end{aligned}$$

Let \mathfrak{g}_1 be a representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and let $\mathcal{D}_{1,1}$ be the vector representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Using the first equation given in (4.5), since \mathfrak{g}_1 is a reducible representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathcal{S}^3(\mathfrak{g}_1)$ reduces to three types of terms (i) $\mathcal{S}^3(\mathcal{D})$, (ii) $\mathcal{S}^2(\mathcal{D}) \otimes \mathcal{D}'$ and (iii) $\mathcal{D} \otimes \mathcal{D}' \otimes \mathcal{D}''$ with $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ three irreducible representations. Thus all possible $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings are of the type (i) $\mathcal{S}^3(\mathcal{D}) \rightarrow \mathcal{D}_{1,1}$, (ii) $\mathcal{S}^2(\mathcal{D}) \otimes \mathcal{D}' \rightarrow \mathcal{D}_{1,1}$ and (iii) $\mathcal{D} \otimes \mathcal{D}' \otimes \mathcal{D}'' \rightarrow \mathcal{D}_{1,1}$. We now characterise more precisely these mappings.

Explicit description of the $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ equivariant mappings

(i) Type I. $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings: $\mathcal{S}^3(\mathcal{D}) \rightarrow \mathcal{D}_{1,1}$

Let $\mathcal{D} = \mathcal{D}_a \otimes \mathcal{D}_b$ with $a, b \in \mathbb{N}$, $\mathcal{D}_{1,1} \subseteq \mathcal{D}_{a,b} \otimes \mathcal{D}_{a,b} \otimes \mathcal{D}_{a,b}$ if a and b odd. From the third equation of (4.5) $\mathcal{D}_{1,1} \subseteq \mathcal{S}^3(\mathcal{D}_{a,b})$ if either

$$I_S : \mathcal{D}_1 \subseteq \mathcal{S}^3(\mathcal{D}_a) \text{ and } \mathcal{D}_1 \subseteq \mathcal{S}^3(\mathcal{D}_b);$$

$$I_A : \mathcal{D}_1 \subseteq \Lambda^3(\mathcal{D}_a) \text{ and } \mathcal{D}_1 \subseteq \Lambda^3(\mathcal{D}_b);$$

$$I_M : \mathcal{D}_1 \subseteq \$\begin{array}{|c|c|} \hline & \\ \hline \end{array}(\mathcal{D}_a) \text{ and } \mathcal{D}_1 \subseteq \$\begin{array}{|c|c|} \hline & \\ \hline \end{array}(\mathcal{D}_b).$$

We call these $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings, mappings of type I_S (symmetric), I_A (antisymmetric) and I_M (mixed) respectively. In particular, when $\mathcal{D} = \mathcal{D}_{a,a}$ and a odd, the mapping $\mathcal{S}^3(\mathcal{D}_{a,a}) \longrightarrow \mathcal{D}_{1,1}$ is *always* $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant and is called type I_{0S}, I_{0A}, I_{0M} respectively. The extension of the Poincaré algebra given in Example 2.5 is of type I_{0M} with $\mathcal{D} = \mathcal{D}_{1,1}$.

(ii) Type II. $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings: $\mathcal{S}^2(\mathcal{D}) \otimes \mathcal{D}' \longrightarrow \mathcal{D}_{1,1}$

Let $\mathcal{D} = \mathcal{D}_{a,b}$ and $\mathcal{D}' = \mathcal{D}_{c,d}$, $\mathcal{D}_{1,1} \subseteq \mathcal{D}_{a,b} \otimes \mathcal{D}_{a,b} \otimes \mathcal{D}_{c,d}$ if c, d odd, and there exists an n such that $2a - 2n - c = 1$ or $c - 2a + 2n = 1$ (and similar relations for b, d). From the second equation of (4.5) $\mathcal{D}_{1,1} \subseteq \mathcal{S}^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{c,d}$ if either

$$II_S : \mathcal{D}_1 \subseteq \mathcal{S}^2(\mathcal{D}_a) \otimes \mathcal{D}_c \text{ and } \mathcal{D}_1 \subseteq \mathcal{S}^2(\mathcal{D}_b) \otimes \mathcal{D}_d;$$

$$II_A : \mathcal{D}_1 \subseteq \Lambda^2(\mathcal{D}_a) \otimes \mathcal{D}_c \text{ and } \mathcal{D}_1 \subseteq \Lambda^2(\mathcal{D}_b) \otimes \mathcal{D}_d.$$

We call these $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings, mappings of type II_S and II_A respectively. In particular, when a, b even $\mathcal{D}_{0,0} \subseteq \mathcal{S}^2(\mathcal{D}_{a,b})$ and $\mathcal{D}_{1,1} \subseteq \mathcal{S}^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{1,1}$ (type II_{0S}). The extension of the Poincaré algebra (5.6) is of type II_{0S} with $\mathcal{D}' = \mathcal{D}_{2,0} \oplus \mathcal{D}_{0,2}, \mathcal{D} = \mathcal{D}_{1,1}$.

(iii) Type III. $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings: $\mathcal{D} \otimes \mathcal{D}' \otimes \mathcal{D}'' \longrightarrow \mathcal{D}_{1,1}$.

Let $\mathcal{D} = \mathcal{D}_{a,b}, \mathcal{D}' = \mathcal{D}_{c,d}$ and $\mathcal{D}'' = \mathcal{D}_{e,f}$ with $a \geq c \geq e$, $\mathcal{D}_{1,1} \subseteq \mathcal{D}_{a,b} \otimes \mathcal{D}_{c,d} \otimes \mathcal{D}_{e,f}$ if $a + c + e$ and $b + d + f$ are odd and if there exists an n such that $a + c + e - 2n = 1$ or $e - a - c + 2n = 1$ (plus similar relations for b, d, f). There are many $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings of these types.

We now give explicit examples of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings of type I, II and III .

Example 4.4 Let $\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{D}_1 = \mathcal{D}_{1,0} \otimes \mathcal{D}_{0,1}$. Using conventional notations for spinors, let $\mathcal{D}_{1,0} = \langle \psi_\alpha, \alpha = 1, 2 \rangle$ and $\mathcal{D}_{0,1} = \langle \bar{\chi}^{\dot{\alpha}}, \dot{\alpha} = 1, 2 \rangle$ be the spinor representations of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Introduce the Dirac Γ -matrices $\{\Gamma_m, \Gamma_n\} = \Gamma_m \Gamma_n + \Gamma_n \Gamma_m = 2\eta_{mn} I_4$, with I_4 the four dimensional identity matrix)

$$\Gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \bar{\sigma}_m & 0 \end{pmatrix},$$

where $\sigma_0 = \bar{\sigma}_0$ is the identity matrix and $\bar{\sigma}_i = -\sigma_i, i = 1, 2, 3$ with σ_i the Pauli matrices. The index structure of the σ_m -matrices is as follow $\sigma_m \rightarrow \sigma_{m\alpha\dot{\alpha}}, \bar{\sigma}_m \rightarrow \bar{\sigma}_m^{\dot{\alpha}\alpha}$. We also define $\psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta, \psi^\alpha = \varepsilon^{\alpha\beta} \psi_\beta, \bar{\chi}_{\dot{\alpha}} = \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}}, \bar{\chi}^{\dot{\alpha}} = \bar{\varepsilon}^{\dot{\alpha}\dot{\beta}} \bar{\chi}_{\dot{\beta}}$ with the invariant antisymmetric $\mathfrak{sl}(2)$ matrices $\varepsilon, \bar{\varepsilon}$ given by $\varepsilon_{12} = \bar{\varepsilon}_{\dot{1}\dot{2}} = -1, \varepsilon^{12} = \bar{\varepsilon}^{\dot{1}\dot{2}} = 1$. A direct calculation gives $\bar{\sigma}_m^{\dot{\beta}\beta} = \varepsilon^{\beta\alpha} \bar{\varepsilon}^{\dot{\beta}\dot{\alpha}} \sigma_{m\alpha\dot{\alpha}}$. Furthermore since the Dirac Γ -matrices are representations of the Clifford algebra, we have the relations $\sigma_m \bar{\sigma}_n + \sigma_n \bar{\sigma}_m = \eta_{mn} \sigma_0$, and thus $\sigma_{m\alpha\dot{\alpha}} \bar{\sigma}_n^{\dot{\alpha}\alpha} = \text{Tr} \sigma_m \bar{\sigma}_n = 2\eta_{mn}$.

Now, we consider the representation

$$\mathcal{D}'_{1,0} \cong \$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}(\mathcal{D}_{1,0}).$$

We introduce the projector (Young symmetriser)

$$P_{\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}} = \frac{1}{3}(1 - (12) + (13) - (123))$$

with

$$(a \ b) = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, (a \ b \ c) = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

two cycles of \mathcal{S}_3 the group of permutation with three elements. A direct calculation gives

$$P_{\begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix}} (\psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma) = \varepsilon_{\alpha\beta} \lambda_\gamma$$

and $\mathcal{D}'_{1,0} = \langle \lambda_\alpha, \alpha = 1, 2 \rangle$ (the same result can be obtained using the usual calculus of the Clebsch-Gordan coefficients). Proceeding along the same lines with $\mathcal{D}_{0,1}$ and introducing

$$\mathcal{D}'_{0,1} \cong \$\begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix} (\mathcal{D}_{0,1}) = \langle \bar{\rho}^{\dot{\alpha}}, \dot{\alpha} = 1, 2 \rangle$$

we obtain

$$P_{\begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix}} (\psi_\alpha \otimes \psi_\beta \otimes \psi_\gamma) \otimes P_{\begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix}} (\bar{\chi}_{\dot{\alpha}} \otimes \bar{\chi}_{\dot{\beta}} \otimes \bar{\chi}_{\dot{\gamma}}) = \varepsilon_{\alpha\beta} \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}} \lambda_\gamma \otimes \bar{\rho}_{\dot{\gamma}}.$$

Symmetrising the R.H.S. we then get

$$\mathcal{S}^3((\psi_\alpha \otimes \bar{\chi}_{\dot{\alpha}}) \otimes (\psi_\beta \otimes \bar{\chi}_{\dot{\beta}}) \otimes (\psi_\gamma \otimes \bar{\chi}_{\dot{\gamma}})) = \varepsilon_{\alpha\beta} \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}} \lambda_\gamma \bar{\rho}_{\dot{\gamma}} + \varepsilon_{\gamma\alpha} \bar{\varepsilon}_{\dot{\gamma}\dot{\alpha}} \lambda_\beta \bar{\rho}_{\dot{\beta}} + \varepsilon_{\beta\gamma} \bar{\varepsilon}_{\dot{\beta}\dot{\gamma}} \lambda_\alpha \bar{\rho}_{\dot{\alpha}}. \quad (4.6)$$

Now, from the isomorphism of $\mathcal{D}_{1,0} \otimes \mathcal{D}_{0,1}$ with the vector representation, and using the relation $\sigma_{m\alpha\dot{\alpha}} \bar{\sigma}_n^{\dot{\alpha}\alpha} = 2\eta_{mn}$ we have the correspondence

$$\begin{aligned} V_m &= \bar{\sigma}_m^{\dot{\alpha}\alpha} \psi_\alpha \otimes \bar{\chi}_{\dot{\alpha}}, & \psi_\alpha \otimes \bar{\chi}_{\dot{\alpha}} &= \frac{1}{2} \sigma^m_{\alpha\dot{\alpha}} V_m, \\ P_m &= \bar{\sigma}_m^{\dot{\alpha}\alpha} \lambda_\alpha \otimes \bar{\rho}_{\dot{\alpha}}, & \lambda_\alpha \otimes \bar{\rho}_{\dot{\alpha}} &= \frac{1}{2} \sigma^m_{\alpha\dot{\alpha}} V_m, \end{aligned} \quad (4.7)$$

(thus $\langle P_m, m = 0, \dots, 3 \rangle \sim \mathcal{D}_{1,1}$, $\langle V_m, m = 0, \dots, 3 \rangle \sim \mathcal{D}'_{1,1}$) and equations (4.6) reduce to

$$\mathcal{S}^3(V_m \otimes V_n \otimes V_p) = \eta_{mn} P_p + \eta_{np} P_m + \eta_{pm} P_n.$$

We denote ${}_3\mathfrak{iso}(1, 3, \mathbb{C}) = \mathfrak{iso}(1, 3, \mathbb{C}) \oplus \mathcal{D}_{1,1}$ the corresponding Lie algebra of order 3. If we now take the real form of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ corresponding to $\mathfrak{sl}(2, \mathbb{C})$ (the universal covering group of $SO(1, 3)$ being $SL(2, \mathbb{C})$), the representation $\mathcal{D}_{1,0}$ and $\mathcal{D}_{0,1}$ become complex conjugate. Thus if we take $\bar{\chi}_{\dot{\alpha}} = (\psi_\alpha)^*$ (the complex conjugate of ψ_α), and similarly $\bar{\rho}_{\dot{\alpha}} = (\lambda_\alpha)^*$, V_m and P_m become real vectors of $\mathfrak{so}(1, 3)$.

Example 4.5 Let $\mathcal{D} = \mathcal{D}_{3,3}$. Using spinor notations

$$\mathcal{D}_{3,0} = \langle \psi_{\alpha\beta\gamma}, \alpha, \beta, \gamma = 1, 2 \rangle, \quad \mathcal{D}_{0,3} = \langle \bar{\chi}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}, \dot{\alpha}, \dot{\beta}, \dot{\gamma} = 1, 2 \rangle,$$

with $\psi_{\alpha\beta\gamma}, \bar{\chi}^{\dot{\alpha}\dot{\beta}\dot{\gamma}}$ symmetric spinor-tensors. This case is more involved than the previous one, because

$$\$ \begin{smallmatrix} \boxed{1} & \boxed{3} \\ \boxed{2} \end{smallmatrix} (\mathcal{D}_{3,0})$$

is a reducible representation. However, using the correspondence (4.7) elements of $\mathcal{D}_{3,3}$ are symmetric traceless tensors of order three

$$T_{mnp} = \bar{\sigma}_m^{\alpha\dot{\alpha}} \bar{\sigma}_n^{\beta\dot{\beta}} \bar{\sigma}_p^{\gamma\dot{\gamma}} \psi_{\alpha\beta\gamma} \bar{\chi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}$$

(the symmetry of T comes from the symmetry of ψ and $\bar{\chi}$ and $T_{mnp} \eta^{nm} = 0$ from $\sigma_{m\alpha\dot{\alpha}} \sigma^m_{\beta\dot{\beta}} = 2\varepsilon_{\alpha\beta} \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}}$). Now it is easy to see that the mapping

$$T_{m_1 n_1 p_1} \otimes T_{m_2 n_2 p_2} \otimes T_{m_3 n_3 p_3} \longrightarrow \eta_{m_1 m_2} \eta_{n_1 n_2} \eta_{p_1 p_3} \eta_{p_2 n_3} P_{p_3}$$

is $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant. Thus, symmetrising the R.H.S we have

$$\begin{aligned} \mathcal{S}^3(T_{m_1 n_1 p_1} \otimes T_{m_2 n_2 p_2} \otimes T_{m_3 n_3 p_3}) = \\ \eta_{m_1 m_2} \eta_{n_1 n_2} \eta_{p_1 m_3} \eta_{p_2 n_3} P_{p_3} + \eta_{m_2 m_3} \eta_{n_2 n_3} \eta_{p_2 m_1} \eta_{p_3 n_1} P_{p_1} + \eta_{m_3 m_1} \eta_{n_3 n_1} \eta_{p_3 m_2} \eta_{p_1 n_2} P_{p_2}. \end{aligned} \quad (4.8)$$

Example 4.6 Let $\mathcal{D} = \mathcal{D}_{1,1} \oplus \mathcal{D}_{2,0} \oplus \mathcal{D}_{0,2}$, we have $\mathbb{C} \subseteq \mathcal{S}^2(\mathcal{D}_{2,0})$, $\mathbb{C} \subseteq \mathcal{S}^2(\mathcal{D}_{0,2})$. Let $\mathcal{D}_{2,0} \oplus \mathcal{D}_{0,2} = \langle A_i, i = 1, 2, 3 \rangle \oplus \langle \bar{A}_i, i = 1, 2, 3 \rangle$. From $\mathcal{S}^2(A_i \otimes A_j) = \delta_{ij}$ and $\mathcal{S}^2(\bar{A}_i \otimes \bar{A}_j) = \delta_{ij}$ the mappings $\mathcal{S}^3(\mathcal{D}_{1,1} \oplus \mathcal{D}_{2,0} \oplus \mathcal{D}_{0,2}) \supseteq (\mathcal{S}^2(\mathcal{D}_{2,0}) \oplus \mathcal{S}^2(\mathcal{D}_{0,2})) \otimes \mathcal{D}_{1,1} \longrightarrow \mathcal{D}_{1,1}$ follows immediately. This gives the trilinear brackets obtained in (5.6).

Example 4.7 Let $\mathcal{D} = \mathcal{D}_{1,0} \oplus \mathcal{D}_{0,1} \oplus \mathcal{D}_{0,0}$. The mapping $\mathcal{S}^3(\mathcal{D}_{1,0} \oplus \mathcal{D}_{0,1} \oplus \mathcal{D}_{0,0}) \supset \mathcal{D}_{1,0} \oplus \mathcal{D}_{0,1} \oplus \mathcal{D}_{0,0} \longrightarrow \mathcal{D}_{1,1}$ is immediate. Such a mapping was considered in [26].

4.3 Lie algebras of order 3 associated to the Poincaré algebra

Let us denote \mathcal{V} the vector space isomorphic to $\mathcal{D}_{1,1}$ generated by the vectors $P_m, m = 0, \dots, 3$. Now, from the characterisation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings from $\mathcal{S}^3(\mathcal{D})$ into $\mathcal{D}_{1,1}$ and lemma 4.1, we construct Lie algebras of order 3 whose zero graded part is isomorphic to the Poincaré algebra.

Theorem 4.8 *Let \mathcal{D} be a reducible representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ such that:*

1. *the action of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ on \mathcal{D} extends to an action of $\mathfrak{iso}(1, 3, \mathbb{C})$ on \mathcal{D} as in Lemma 4.1;*
2. *there exist an $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mapping from $\mathcal{S}^3(\mathcal{D}) \longrightarrow \mathcal{V}$.*

Then if $\mathfrak{g} = \mathfrak{iso}(1, 3, \mathbb{C}) \oplus \mathcal{D}$ is a Lie algebra of order 3, the action of \mathcal{V} on \mathcal{D} is trivial.

Proof.

Type I : Assume $\mathcal{S}^3(\mathcal{D}_{a,b}) \longrightarrow \mathcal{V}$. This means that a, b are odd.

1. Suppose there exists a representation $\mathcal{D}' \subseteq \mathcal{D}$ (not necessarily irreducible) such that $\mathcal{D}' \subseteq [\mathcal{V}, \mathcal{D}_{a,b}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y_3 = Y_4 \in \mathcal{D}_{a,b}$ leads to a contradiction, thus $[\mathcal{V}, \mathcal{D}_{a,b}] = 0$.
2. Suppose there exist a representation $\mathcal{D}_{c,d} \subseteq \mathcal{D}$ such that $\mathcal{D}_{a,b} \subseteq [\mathcal{V}, \mathcal{D}_{c,d}]$. Since a, b are odd, c, d are even and thus $\mathcal{V} \not\subseteq \mathcal{S}^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{c,d}$. The Jacobi identity J4 with $Y_1, Y_2, Y_3 \in \mathcal{D}_{a,b}, Y_4 \in \mathcal{D}_{c,d}$ together with $\mathcal{S}^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{c,d} = 0$ gives a contradiction and $[\mathcal{V}, \mathcal{D}_{c,d}] = 0$.

Type II : Assume $\mathcal{V} \subseteq \mathcal{S}^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{c,d}$ and $\mathcal{S}^3(\mathcal{D}_{a,b}) = \mathcal{S}^3(\mathcal{D}_{c,d}) = 0$ (not of type I). In this case c, d are odd.

1. Suppose $\mathcal{D}_{c,d} \subseteq [\mathcal{V}, \mathcal{D}_{a,b}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y_3 \in \mathcal{D}_{a,b}, Y_4' \in \mathcal{D}_{c,d}$ gives a contradiction and $[\mathcal{V}, \mathcal{D}_{a,b}] = 0$.
2. Suppose $\mathcal{D}_{a,b} \subseteq [\mathcal{V}, \mathcal{D}_{c,d}]$, thus a, b are even and $\mathcal{V} \not\subseteq \mathcal{S}^2(\mathcal{D}_{c,d}) \otimes \mathcal{D}_{a,b}$. The Jacobi identity J4 with $Y_1 = Y_2 \in \mathcal{D}_{a,b}, Y_3' = Y_4' \in \mathcal{D}_{c,d}$ gives a contradiction and $[\mathcal{V}, \mathcal{D}_{a,b}] = 0$.
3. Suppose there exists $\mathcal{D}_{e,f} \subseteq \mathcal{D}$, with $\mathcal{D}_{e,f} \neq \mathcal{D}_{c,d}$, such that $\mathcal{D}_{e,f} \subseteq [\mathcal{V}, \mathcal{D}_{a,b}]$. The same argument as in the point 1. above gives $[\mathcal{V}, \mathcal{D}_{a,b}] = 0$.
4. Suppose there exists $\mathcal{D}_{e,f} \subseteq \mathcal{D}$, with $\mathcal{D}_{e,f} \neq \mathcal{D}_{a,b}$, such that $\mathcal{D}_{e,f} \subseteq [\mathcal{V}, \mathcal{D}_{c,d}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y \in \mathcal{D}_{ab}$ and $Y_3' = Y_4' = Y' \in \mathcal{D}_{cd}$ gives $[Y, \{Y, Y', Y'\}] + [Y', \{Y, Y, Y'\}] = 0$. If we suppose that $\{Y, Y', Y'\} = P \in \mathcal{V}$ we know from the points 1. and 3. above that $[P, Y] = 0$. Thus the previous identity becomes $[Y', \{Y, Y, Y'\}] = [Y', P] = 0$ and $[\mathcal{V}, \mathcal{D}_{c,d}] = 0$.
5. Suppose there exists $\mathcal{D}_{e,f} \subseteq \mathcal{D}$, with $\mathcal{D}_{e,f} \neq \mathcal{D}_{a,b}$ and $\mathcal{D}_{e,f} \neq \mathcal{D}_{c,d}$ such that either $\mathcal{D}_{a,b} \subseteq [\mathcal{V}, \mathcal{D}_{e,f}]$ or $\mathcal{D}_{c,d} \subseteq [\mathcal{V}, \mathcal{D}_{e,f}]$. The Jacobi identity J4 with $Y_1 = Y_2 = Y \in \mathcal{D}_{a,b}, Y' \in \mathcal{D}_{c,d}, Y'' \in \mathcal{D}_{e,f}$ gives $2[Y, \{Y, Y', Y''\}] + [Y', \{Y, Y, Y''\}] + [Y'', \{Y, Y, Y'\}] = 0$. If $\{Y, Y', Y''\} \in \mathcal{V}$ or $\{Y, Y, Y''\} \in \mathcal{V}$ since $[\mathcal{V}, \mathcal{D}_{a,b}] = [\mathcal{V}, \mathcal{D}_{c,d}] = 0$ (see 1., 2., 3. and 4. above), the previous identity reduces to $[Y'', \{Y, Y, Y'\}] = [Y'', P] = 0$ and thus $[\mathcal{V}, \mathcal{D}_{e,f}] = 0$.

Type III : Assume $\mathcal{V} \subseteq \mathcal{D}_{a,b} \otimes \mathcal{D}_{c,d} \otimes \mathcal{D}_{e,f}$, $\mathcal{S}^3(\mathcal{D}_{a,b}) = \mathcal{S}^3(\mathcal{D}_{c,d}) = \mathcal{S}^3(\mathcal{D}_{e,f}) = 0$ (not of type I) and $\mathcal{S}^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{c,d} = \mathcal{S}^2(\mathcal{D}_{a,b}) \otimes \mathcal{D}_{e,f} = \mathcal{S}^2(\mathcal{D}_{c,d}) \otimes \mathcal{D}_{a,b} = \mathcal{S}^2(\mathcal{D}_{c,d}) \otimes \mathcal{D}_{e,f} = \mathcal{S}^2(\mathcal{D}_{e,f}) \otimes \mathcal{D}_{a,b} = \mathcal{S}^2(\mathcal{D}_{e,f}) \otimes \mathcal{D}_{c,d} = 0$ (not of type II).

1. If we assume $[\mathcal{V}, \mathcal{D}_{a,b}] \subseteq \mathcal{D}_{g,h}$ with $\mathcal{D}_{g,h} = \mathcal{D}_{c,d}$ or $\mathcal{D}_{g,h} \neq \mathcal{D}_{c,d}, \mathcal{D}_{g,h} \neq \mathcal{D}_{e,f}$, the Jacobi identity J4 with $Y_1, Y_2 \in \mathcal{D}_{a,b}, Y_3' \in \mathcal{D}_{c,d}, Y_4'' \in \mathcal{D}_{e,f}$ leads to a contradiction and $[\mathcal{V}, \mathcal{D}_{a,b}] = 0$.
2. Suppose there exists $\mathcal{D}_{g,h}$ such that $[\mathcal{V}, \mathcal{D}_{g,h}] \not\subseteq \mathcal{D}_{a,b}$, the Jacobi identity J4 with $Y_1 \in \mathcal{D}_{a,b}, Y_2 \in \mathcal{D}_{c,d}, Y_3 \in \mathcal{D}_{e,f}$ and $Y_4 \in \mathcal{D}_{g,h}$ leads to a contradiction and thus $[\mathcal{V}, \mathcal{D}_{g,h}] = 0$.

This means that the action of \mathcal{V} on \mathcal{D} is trivial and thus $[\mathcal{V}, \mathcal{D}] = 0$. The remaining Jacobi identities are easy to be checked. Which ends the proof. QED.

Corollary 4.9 *With the hypothesis of theorem 4.8 the action of \mathcal{V} on \mathcal{D} is trivial i.e. $[\mathcal{V}, \mathcal{D}] = 0$.*

Remark 4.10 *Differently as in theorem 4.8 if $\mathfrak{g} = (\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathcal{V}) \oplus \mathcal{D}$ is a Lie algebra of order 3 satisfying $[\mathcal{V}, \mathcal{D}] = 0$ then $\mathcal{S}^3(\mathcal{D}) \longrightarrow \mathcal{V}$. Indeed if we suppose for contradiction that $\mathcal{S}^3(\mathcal{D}) \longrightarrow \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, and let $Y_1, Y_2, Y_3 \in \mathcal{D}$ such that $\{Y_1, Y_2, Y_3\} = aL$ with $L \in \mathfrak{sl}(2) \oplus \mathfrak{sl}(2), a \in \mathbb{C}$. The Jacobi identity J3 with $P \in \mathcal{V}$ and Y_1, Y_2, Y_3 as above leads to $a = 0$ since the elements of \mathcal{D} commute with the elements of \mathcal{V} and L do not commute with P .*

The following property classify all Lie algebras of order 3 based on the Poincaré algebra such that the representation \mathcal{D} is of dimension 4.

Proposition 4.11 *Let $\mathfrak{g} = \mathfrak{iso}(1, 3, \mathbb{C}) \oplus \mathcal{D}$ be an elementary Lie algebra of order 3, with \mathcal{D} a representation of dimension 4. Then,*

1. $\mathcal{D} \cong \mathcal{D}_{1,1}$;
2. $[\mathcal{V}, \mathcal{D}_{1,1}] = 0$;
3. $\mathfrak{g} \cong {}_3\mathfrak{iso}(1, 3, \mathbb{C})$ (the complexified of the Lie algebra of order three of Example 2.5).

Proof. 1. Since the representation $\mathcal{D}_\ell, \ell \in \mathbb{N}$ of $\mathfrak{sl}(2)$ is of dimension $\ell+1$, the four dimensional representations of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ are (up a permutation of the action $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$) :

$$\mathcal{D}_{3,0}, \mathcal{D}_{2,0} \oplus \mathcal{D}_{0,0}, \mathcal{D}_{1,1}, \mathcal{D}_{1,0} \oplus \mathcal{D}_{0,1}, \mathcal{D}_{1,0} \oplus \mathcal{D}_{0,0} \oplus \mathcal{D}_{0,0}, \mathcal{D}_{0,0} \oplus \mathcal{D}_{0,0} \oplus \mathcal{D}_{0,0} \oplus \mathcal{D}_{0,0}.$$

Since $\mathcal{S}^3(\mathcal{D}) \longrightarrow \mathfrak{iso}(1, 3, \mathbb{C})$, a simple weights argument shows that the only possibilities are (i) $\mathcal{D} = \mathcal{D}_{1,1}$ with $\mathcal{V} \subseteq \mathcal{S}^3(\mathcal{D})$ (in agreement with the explicit description of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ -equivariant mappings) or (ii) by Example 2.4 $\mathfrak{sl}(2) \cong \mathcal{D}_{2,0} \subseteq \mathcal{S}^3(\mathcal{D}_{2,0})$. In the second case we have by Remark 4.3 $[\mathcal{V}, \mathcal{D}_{2,0}] = 0$ and thus the Jacobi identity J3 with $P \in \mathcal{V}, Y_1, Y_2, Y_3 \in \mathcal{D}_{2,0}$ leads to a contradiction since P commute with Y_i and not commute with U (see Eq.(4.3)). Thus, the only non-trivial Lie algebra of order 3 is then constructed with $\mathcal{D}_{1,1}$.

2. Since \mathcal{D} is an irreducible representation of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, by Remark 4.3, the action of \mathcal{V} on \mathcal{D} is trivial and $[\mathcal{V}, \mathcal{D}] = 0$.

3. Since $[\mathcal{V}, \mathcal{D}] = 0$, by Corollary 4.10, we have $\mathcal{V} \subset \mathcal{S}^3(\mathcal{D})$ (a simple weights argument, as we have seen in 1. above, also show that $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \not\subseteq \mathcal{S}^3(\mathcal{D})$). This means, introducing $v_{++}, v_{+-}, v_{-+}, v_{--}$ a basis of \mathcal{D} (with notations similar to (4.2)), and using a simple weights argument, that the only non-trivial trilinear brackets are:

$$\begin{aligned} \{v_{++}, v_{++}, v_{--}\} &= \alpha_1 P_{++}, & \{v_{--}, v_{--}, v_{++}\} &= \beta_1 P_{--}, \\ \{v_{++}, v_{+-}, v_{-+}\} &= \alpha_2 P_{++}, & \{v_{--}, v_{-+}, v_{+-}\} &= \beta_2 P_{--}, \\ \{v_{++}, v_{+-}, v_{--}\} &= \gamma_1 P_{+-}, & \{v_{--}, v_{-+}, v_{++}\} &= \delta_1 P_{-+}, \\ \{v_{+-}, v_{+-}, v_{-+}\} &= \gamma_2 P_{+-}, & \{v_{-+}, v_{-+}, v_{+-}\} &= \delta_3 P_{-+}. \end{aligned}$$

The action of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ on \mathcal{D} is given by (see (4.3))

$$\begin{aligned} [U_+, v_{-\varepsilon}] &= v_{+\varepsilon}, & [V_+, v_{\varepsilon-}] &= -v_{\varepsilon+}, \\ [U_-, v_{+\varepsilon}] &= v_{-\varepsilon}, & [V_-, v_{\varepsilon+}] &= -v_{\varepsilon-}, \\ [U_0, v_{\varepsilon\varepsilon'}] &= \varepsilon v_{\varepsilon\varepsilon'}, & [V_0, v_{\varepsilon\varepsilon'}] &= \varepsilon' v_{\varepsilon\varepsilon'}, \end{aligned}$$

with $\varepsilon, \varepsilon' = \pm$. The Jacobi identity J3 gives

$$\alpha_2 = -\frac{1}{2}\alpha_1, \beta_1 = \alpha_1, \beta_2 = -\frac{1}{2}\alpha_1, \gamma_1 = \frac{1}{2}\alpha_1, \gamma_2 = -\alpha_1, \delta_1 = \frac{1}{2}\alpha_1, \delta_2 = -\alpha_1.$$

If $\alpha_1 \neq 0$, we set

$$\begin{aligned} v_0 &= -\sqrt[3]{\frac{1}{2\alpha_1}}(v_{+-} + v_{-+}), & v_3 &= -\sqrt[3]{\frac{1}{2\alpha_1}}(v_{+-} - v_{-+}), \\ v_1 &= -\sqrt[3]{\frac{1}{2\alpha_1}}(v_{++} + v_{--}), & v_2 &= i\sqrt[3]{\frac{1}{2\alpha_1}}(v_{++} - v_{--}), \end{aligned}$$

and we get

$$\{v_\mu, v_\nu, v_\rho\} = \eta_{\mu\nu}P_\rho + \eta_{\mu\rho}P_\nu + \eta_{\nu\rho}P_\mu.$$

So, the algebra is isomorphic to the complexified elementary Lie algebra of order 3 of Example 2.5. This results remains true if we consider its real form corresponding to the Lie algebra of order 3 of example 2.5. In this case if $\alpha_1 > 0$ we rescale the coefficients by $-\sqrt[3]{\frac{1}{2\alpha_1}}$ as above, and when $\alpha_1 < 0$ we rescale the coefficients by $\sqrt[3]{\frac{-1}{2\alpha_1}}$. Q.E.D.

5 Contractions of the Poincaré-algebra of order 3

5.1 Contractions of elementary Lie algebras of order 3

The variety $\mathcal{F}_{m,n}$ being an algebraic variety, one can naturally endow it with the Zariski topology.

Definition 5.1 *A contraction of φ is a point $\varphi' \in \mathcal{F}_{m,n}$ such that $\varphi' \in \overline{\mathcal{O}}_\varphi$, the closure in the Zariski sense.*

In the complex case, the notion of contraction is equivalent to the following. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be a given multiplication of elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and let $(h_p)_{p \in \mathbb{N}}$ (with $h_p = (h_{0,p}, h_{1,p}) \in GL(m, n)$) be a sequence of isomorphisms. Define $\varphi_p = (\varphi_{1,p}, \varphi_{2,p}, \varphi_{3,p})$ by

$$\begin{aligned} \varphi_{1,p}(X_1, X_2) &= h_{0,p}^{-1} \varphi_1(h_{0,p}(X_1), h_{0,p}(X_2)), \\ \varphi_{2,p}(X_1, Y_2) &= h_{1,p}^{-1} \varphi_2(h_{0,p}(X_1), h_{1,p}(Y_2)), \\ \varphi_{3,p}(Y_1, Y_2, Y_3) &= h_{0,p}^{-1} \varphi_3(h_{1,p}(Y_1), h_{1,p}(Y_2), h_{1,p}(Y_3)). \end{aligned} \tag{5.1}$$

If the limit $\lim_{p \rightarrow +\infty} \varphi_p$ exists, this limit is in the closure of \mathcal{O}_φ . Then it is a contraction of φ . Note that, in the complex case, every contraction of φ is obtained by this way [11].

Moreover, Inönü-Wigner contractions [16] turn out to be a relevant subclass of contractions. We consider the automorphisms $h_\varepsilon = (h_{0,\varepsilon}, h_{1,\varepsilon})$ of the form $h_{0,\varepsilon} = h_0^{(1)} + \varepsilon h_0^{(2)}$ and $h_{1,\varepsilon} = h_1^{(1)} + \varepsilon h_1^{(2)}$ with $h_0^{(1)}, h_1^{(1)}$ singular, $h_0^{(2)}, h_1^{(2)}$ regular and ε infinitesimal. A further particularisation inspired from the Weimar-Woods construction [30] is given by

$$h_\varepsilon = \text{diag}(\varepsilon^{a_1}, \dots, \varepsilon^{a_m}, \varepsilon^{b_1}, \dots, \varepsilon^{b_n})$$

with $a_i, b_j \in \mathbb{Z}$ ($i = 1, \dots, m, j = 1, \dots, n$). Hence $h_{0,\varepsilon}(X_i) = \varepsilon^{a_i} X_i$ and $h_{0,\varepsilon}(Y_j) = \varepsilon^{a_j} Y_j$ and (5.1) become

$$\begin{aligned} \varphi_{1,\varepsilon}(X_i, X_j) &= \varepsilon^{a_i+a_j-a_k} C_{ij}^k X_k, \\ \varphi_{2,\varepsilon}(X_i, Y_j) &= \varepsilon^{a_i+b_j-b_k} D_{ij}^k Y_k, \\ \varphi_{3,\varepsilon}(Y_i, Y_j, Y_k) &= \varepsilon^{b_i+b_j+b_k-a_\ell} E_{ijk}^\ell X_\ell. \end{aligned} \tag{5.2}$$

As already stated, one can define a contraction if the limit $\varepsilon \rightarrow 0$ exists, i.e. if $a_i + a_j - a_k \geq 0$, $a_i + b_j - b_k \geq 0$ and $b_i + b_j + b_k - a_\ell \geq 0$ for any a and b .

Examples.

Let $\mathcal{F}_{1,1}$ be the algebraic variety of $2 = (1 + 1)$ -dimensional elementary Lie algebras of order 3. We consider a basis $\{X, Y\}$ of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ adapted for this decomposition.

Proposition 5.2 *Any two-dimensional Lie algebra of order 3 $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is isomorphic to one of the following Lie algebras of order 3*

1. \mathfrak{g}_1^3 : $\{Y, Y, Y\} = X$, $[X, Y] = 0$;
2. \mathfrak{g}_2^3 : $[X, Y] = Y$, $\{Y, Y, Y\} = 0$;
3. \mathfrak{g}_3^3 : the trivial Lie algebra of order 3.

Proof. We consider the most general possibility for the structure constants of a two-dimensional elementary Lie algebra of order 3:

$$[X, Y] = \alpha_1 Y, \{Y, Y, Y\} = \alpha_2 X. \quad (5.3)$$

The Jacobi identities J1-J4 imply $\alpha_1 \alpha_2 = 0$, and we obtain

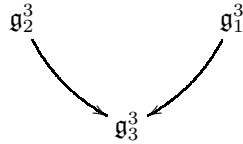
$$\begin{aligned} \mathfrak{g}_1^3, \alpha_1 = 0, \alpha_2 = 1; \\ \mathfrak{g}_2^3, \alpha_1 = 1, \alpha_2 = 0; \\ \mathfrak{g}_3^3, \alpha_1 = 0, \alpha_2 = 0. \end{aligned}$$

QED

Corollary 5.3 *The variety $\mathcal{F}_{1,1}$ of 2-dimensional elementary Lie algebras of order 3, is the union of two irreducible algebraic components U_1 and U_2 with*

$$U_1 = \overline{\mathcal{O}_{\mathfrak{g}_1^3}} \text{ and } U_2 = \overline{\mathcal{O}_{\mathfrak{g}_2^3}}.$$

Proof. One has the following contraction scheme



where by $A \rightarrow B$ we denote a contraction of the algebra A to the algebra B (B is a contraction of A). QED

Remark 5.4 *The algebra \mathfrak{g}_1^3 has been considered in [3, 4, 5, 6, 18, 19].*

5.2 Contraction which leads to a non-trivial extension of the Poincaré algebra

Let $\mathfrak{g} = \mathfrak{so}(2, 3) \oplus \text{ad } \mathfrak{so}(2, 3)$. Using vector indices of $\mathfrak{so}(1, 3)$ coming from the inclusion $\mathfrak{so}(1, 3) \subset \mathfrak{so}(2, 3)$ we introduce $\{M_{mn} = -M_{nm}, M_{m4} = -M_{4m}, m, n = 0, \dots, 3, m < n\}$ a basis of $\mathfrak{so}(2, 3)$ and $\{J_{mn} = -J_{nm}, J_{m4} = -J_{4m}, m, n = 0, \dots, 3, m < n\}$ the corresponding basis of $\text{ad } \mathfrak{so}(2, 3)$. The multiplication law φ of the elementary Lie algebra of order 3 $\mathfrak{so}(2, 3) \oplus \text{ad } \mathfrak{so}(2, 3)$ writes

$$\begin{aligned}
\varphi_1(M_{mn}, M_{pq}) &= -\eta_{nq}M_{mp} - \eta_{mp}M_{nq} + \eta_{mq}M_{np} + \eta_{np}M_{mq}, \\
\varphi_1(M_{mn}, M_{p4}) &= -\eta_{mp}M_{n4} + \eta_{np}M_{m4}, \\
\varphi_1(M_{m4}, M_{p4}) &= -M_{mp}, \\
\varphi_2(M_{mn}, J_{pq}) &= -\eta_{nq}J_{mp} - \eta_{mp}J_{nq} + \eta_{mq}J_{np} + \eta_{np}J_{mq}, \\
\varphi_2(M_{mn}, J_{p4}) &= -\eta_{mp}J_{n4} + \eta_{np}J_{m4}, \\
\varphi_2(M_{m4}, J_{pq}) &= -\eta_{mp}J_{4q} + \eta_{mq}J_{4p}, \\
\varphi_2(M_{m4}, J_{p4}) &= -J_{mp}, \\
\varphi_3(J_{mn}, J_{pq}, J_{rs}) &= (\eta_{mp}\eta_{nq} - \eta_{mq}\eta_{np})M_{rs} + (\eta_{mr}\eta_{ns} - \eta_{ms}\eta_{nr})M_{pq} + (\eta_{pr}\eta_{qs} - \eta_{ps}\eta_{qr})M_{mn}, \\
\varphi_3(J_{mn}, J_{pq}, J_{r4}) &= (\eta_{mp}\eta_{nq} - \eta_{mq}\eta_{np})J_{r4}, \\
\varphi_3(J_{mn}, J_{p4}, J_{r4}) &= \eta_{pr}M_{mn}, \\
\varphi_3(J_{m4}, J_{p4}, J_{r4}) &= \eta_{mp}M_{r4} + \eta_{mr}M_{p4} + \eta_{pr}M_{m4},
\end{aligned} \tag{5.4}$$

where $\eta_{mn} = \text{diag}(1, -1, -1, -1)$. In [26], an Inönü-Wigner contraction was done

$$\begin{aligned}
L_{mn} &= h_0(M_{mn}) = M_{mn}, \\
P_m &= h_0(M_{m4}) = \varepsilon M_{m4}, \\
V_{mn} &= h_1(J_{mn}) = \sqrt[3]{\varepsilon} J_{mn}, \\
V_m &= h_1(J_{m4}) = \sqrt[3]{\varepsilon} J_{m4},
\end{aligned} \tag{5.5}$$

and the limit $\varepsilon \rightarrow 0$ realised the contraction. The contracted Lie algebra of order 3 is given by

$$\begin{aligned}
\varphi_1(L_{mn}, L_{pq}) &= -\eta_{nq}L_{mp} - \eta_{mp}L_{nq} + \eta_{mq}L_{np} + \eta_{np}L_{mq}, \\
\varphi_1(L_{mn}, P_p) &= -\eta_{mp}P_n + \eta_{np}P_m, \\
\varphi_1(P_m, P_p) &= 0, \\
\varphi_2(L_{mn}, V_{pq}) &= -\eta_{nq}V_{mp} - \eta_{mp}V_{nq} + \eta_{mq}V_{np} + \eta_{np}V_{mq}, \\
\varphi_2(L_{mn}, V_p) &= -\eta_{mp}V_n + \eta_{np}V_m, \\
\varphi_2(P_m, V_{pq}) &= 0, \\
\varphi_2(P_m, V_p) &= 0, \\
\varphi_3(V_{mn}, V_{pq}, V_{rs}) &= 0, \\
\varphi_3(V_{mn}, V_{pq}, V_r) &= (\eta_{mp}\eta_{nq} - \eta_{mq}\eta_{np})P_r, \\
\varphi_3(V_{mn}, V_p, V_r) &= 0, \\
\varphi_3(V_m, V_p, V_r) &= \eta_{mp}P_r + \eta_{mr}P_p + \eta_{pr}P_m.
\end{aligned} \tag{5.6}$$

Thus L_{mn}, P_m generate the Poincaré algebra and V_{mn} and V_m are in the adjoint and vector representations of $\mathfrak{so}(1, 3)$.

Remark 5.5 *The subalgebra generated by L_{mn}, P_m and V_m is the algebra of Example 2.5.*

6 Deformations of Lie algebras of order 3

6.1 Definition

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie algebra of order 3 on the complex field (or on a field of characteristic zero). Let A be a commutative local \mathbb{K} -algebra, \mathfrak{g} its maximal ideal. We assume that the residual field is isomorphic to \mathbb{K} . A A -Lie algebra of order F is a Lie algebra of order F whose coefficients in a \mathbb{K} -basis belongs to A . We will assume that A admits an augmentation $\epsilon : A \rightarrow \mathbb{K}$. The ideal $m_\epsilon := \text{Ker } \epsilon$ is the maximal ideal of A . In this context, a deformation λ of \mathfrak{g} with the base (A, m) or simply with the base A , is a A -Lie algebra of order F on the tensor product $A \otimes_{\mathbb{K}} \mathbb{K}^n$ with brackets $[\cdot, \cdot]_\lambda$ and $\{\cdot, \cdot\}_\lambda$ satisfying the Jacobi identities J1-J4.

Sometimes, we add hypothesis on A , for example A is finitely generated or A is noetherian. In this paper, following [14], we assume that A is a valuation algebra. It is a local algebra with the following property : if x belongs to the field of fractions of A but not to A , then the converse x^{-1} belongs to \mathfrak{g} . The most interesting examples of deformations (the formal deformations of Gerstenhaber, the nonstandard perturbations) satisfy this hypothesis. Moreover, a deformation in a valued algebra always admits a finite decomposition, that is is defined by a finite number of ideals of A . Then every deformation has a finite Krull dimension and this means that an hypothesis such as the noetherian property is superfluous.

6.2 The Gerstenhaber products

The Gerstenhaber products have been introduced in [9] and [10] to study the deformations of associative algebras and the structure of Hochschild cohomology groups. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be a given multiplication of elementary Lie algebras of order 3, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The identities J1-J4 are equivalent to

$$\begin{aligned} \varphi_1(\varphi_1(X_1, X_2), X_3) + \varphi_1(\varphi_1(X_3, X_1), X_2) + \varphi_1(\varphi_1(X_2, X_3), X_1) &= 0, \\ \varphi_2(\varphi_1(X_1, X_2), Y) + \varphi_2(\varphi_2(X_2, Y), X_1) + \varphi_2(\varphi_2(Y, X_1), X_2) &= 0, \\ \varphi_1(X, \varphi_3(Y_1, Y_2, Y_3)) - \varphi_3(\varphi_2(X, Y_1), Y_2, Y_3) - \varphi_3(Y_1, \varphi_2(X, Y_2), Y_3) - \varphi_3(Y_1, Y_2, \varphi_2(X, Y_3)) &= 0, \\ \varphi_2(Y_1, \varphi_3(Y_2, Y_3, Y_4)) + \varphi_2(Y_2, \varphi_3(Y_1, Y_3, Y_4)) + \varphi_2(Y_3, \varphi_3(Y_1, Y_2, Y_4)) + \varphi_2(Y_4, \varphi_3(Y_1, Y_2, Y_3)) &= 0. \end{aligned} \quad (6.1)$$

If φ and φ' are two products of elementary Lie algebras of order 3, we can define

$$\begin{aligned} \varphi \circ_1 \varphi' : \quad & (\mathfrak{g}_0 \otimes \mathfrak{g}_0 \otimes \mathfrak{g}_0) \rightarrow \mathfrak{g}_0 \\ & X_1 \otimes X_2 \otimes X_3 \mapsto \varphi_1(\varphi'_1(X_1, X_2), X_3) + \varphi_1(\varphi'_1(X_3, X_1), X_2) + \varphi_1(\varphi'_1(X_2, X_3), X_1), \\ \varphi \circ_2 \varphi' : \quad & (\mathfrak{g}_0 \otimes \mathfrak{g}_0 \otimes \mathfrak{g}_1) \rightarrow \mathfrak{g}_1 \\ & X_1 \otimes X_2 \otimes Y \mapsto \varphi_2(\varphi'_1(X_1, X_2), Y) + \varphi_2(\varphi'_2(X_2, Y), X_1) + \varphi_2(\varphi'_2(Y, X_1), X_2), \\ \varphi \circ_3 \varphi' : \quad & (\mathfrak{g}_0 \otimes \mathcal{S}^3(\mathfrak{g}_1)) \rightarrow \mathfrak{g}_0 \\ & X \otimes (Y_1, Y_2, Y_3) \mapsto \varphi_1(X, \varphi'_3(Y_1, Y_2, Y_3)) - \varphi_3(\varphi'_2(X, Y_1), Y_2, Y_3) \\ & \quad - \varphi_3(Y_1, \varphi'_2(X, Y_2), Y_3) - \varphi_3(Y_1, Y_2, \varphi'_2(X, Y_3)), \\ \varphi \circ_4 \varphi' : \quad & (\mathfrak{g}_1 \otimes \mathcal{S}^3(\mathfrak{g}_1)) \rightarrow \mathfrak{g}_1 \\ & Y_1 \otimes (Y_2, Y_3, Y_4) \mapsto \varphi_2(Y_1, \varphi'_3(Y_2, Y_3, Y_4)) + \varphi_2(Y_2, \varphi'_3(Y_1, Y_3, Y_4)) \\ & \quad + \varphi_2(Y_3, \varphi'_3(Y_1, Y_2, Y_4)) + \varphi_2(Y_4, \varphi'_3(Y_1, Y_2, Y_3)). \end{aligned} \quad (6.2)$$

Proposition 6.1 *The map φ endows \mathfrak{g} with a structure of elementary Lie algebra of order 3 iff*

$$\varphi \circ_i \varphi = 0 \text{ for } i = 1, \dots, 4. \quad (6.3)$$

6.3 Gerstenhaber deformations

In this section we assume that the algebra of valuation A is the algebra $\mathbb{C}[[t]]$ of formal sequence. Its law is written φ_t

$$\varphi_t : A \rightarrow (\mathfrak{g}_0 \oplus \mathfrak{g}_1) \otimes \mathbb{C}[[t]] \quad (6.4)$$

with

$$\varphi_t = \varphi + t\psi^{(1)} + t^2\psi^{(2)} + \dots + t^n\psi^{(n)} + \dots, \quad (6.5)$$

where the $\psi^{(i)}$'s are linear applications from A to \mathfrak{g} , satisfying (6.3).

Proposition 6.2 *Considering a deformation φ_t of φ , the maps $\psi^{(p)}$ (with $p \in \mathbb{N}$) satisfy the equations*

$$\sum_{p+q=r} \psi^{(p)} \circ_i \psi^{(q)} = 0, \text{ for any } i = 1, \dots, 4, r \in \mathbb{N} \quad (6.6)$$

where $\psi^{(0)} = \varphi$.

Proof. As φ_t is a deformation of φ it satisfies $\varphi_t \circ_i \varphi_t = 0$. For $i = 1$, equation (6.6) is just the condition of the deformations of Gerstenhaber for Lie algebras. We explicitly prove (6.6) for $i = 2$ the two remaining cases being similar. If one checks only the terms in t^2 , only the terms $\varphi + t\psi^{(1)} + t^2\psi^{(2)}$ will matter. Inserting

$$\begin{aligned}\varphi_{t1} &= \varphi_1 + t\psi_1^{(1)} + t^2\psi_1^{(2)} \\ \varphi_{t2} &= \varphi_2 + t\psi_2^{(1)} + t^2\psi_2^{(2)} \\ \varphi_{t3} &= \varphi_3 + t\psi_3^{(1)} + t^2\psi_3^{(2)}\end{aligned}\tag{6.7}$$

in (6.1), the coefficient of degree 1 leads to

$$\begin{aligned}\varphi_2(\psi_1^{(1)}(X_1, X_2), Y) + \psi_2^{(1)}(\varphi_1(X_1, X_2), Y) + \varphi_2(\psi_2^{(1)}(X_2, Y), X_1) \\ + \psi_2^{(1)}(\varphi_2(X_2, Y), X_1) + \varphi_2(\psi_2^{(1)}(Y, X_1), X_2) + \psi_2^{(1)}(\varphi_2(Y, X_1), X_2) = 0\end{aligned}\tag{6.8}$$

and the coefficient of degree 2 gives

$$\begin{aligned}\varphi_2(\psi_1^{(2)}(X_1, X_2), Y) + \psi_2^{(2)}(\varphi_1(X_1, X_2), Y) + \psi_2^{(1)}(\psi_1^{(1)}(X_1, X_2), Y) \\ + \varphi_2(\psi_2^{(2)}(X_2, Y), X_1) + \psi_2^{(2)}(\varphi_2(X_2, Y), X_1) + \psi_2^{(1)}(\psi_2^{(1)}(X_2, Y), X_1) \\ + \varphi_2(\psi_2^{(2)}(Y, X_1), X_2) + \psi_2^{(2)}(\varphi_2(Y, X_1), X_2) + \psi_2^{(1)}(\psi_2^{(1)}(Y, X_1), X_2) = 0.\end{aligned}\tag{6.9}$$

Then $\sum_{p+q=1} \psi^{(p)} \circ_2 \psi^{(r)} = 0$ and $\sum_{p+q=2} \psi^{(p)} \circ_2 \psi^{(r)} = 0$. Similarly, one proves (6.6) for any $r \in \mathbb{N}^*$. QED

Definition 6.3 *An infinitesimal deformation of φ is a deformation φ_t of the form*

$$\varphi_t = \varphi + t\psi^{(1)}.$$

Let $\varphi_t = (\varphi_1 + t\psi_1^{(1)}, \varphi_2 + t\psi_2^{(1)}, \varphi_3 + t\psi_3^{(1)})$. Identities (6.3) for the coefficient of t lead to

$$\begin{aligned}\varphi_1(\psi_1^{(1)}(X_1, X_2), X_3) + \psi_1^{(1)}(\varphi_1(X_1, X_2), X_3) + \varphi_1(\psi_1^{(1)}(X_3, X_1), X_2) \\ + \psi_1^{(1)}(\varphi_1(X_3, X_1), X_2) + \varphi_1(\psi_1^{(1)}(X_2, X_3), X_1) + \psi_1^{(1)}(\varphi_1(X_2, X_3), X_1) = 0, \\ \varphi_2(\psi_1^{(1)}(X_1, X_2), Y) + \psi_2^{(1)}(\varphi_1(X_1, X_2), Y) + \varphi_2(\psi_2^{(1)}(X_2, Y), X_1) \\ + \psi_2^{(1)}(\varphi_2(X_2, Y), X_1) + \varphi_2(\psi_2^{(1)}(Y, X_1), X_2) + \psi_2^{(1)}(\varphi_2(Y, X_1), X_2) = 0, \\ \varphi_1(X, \psi_3^{(1)}(Y_1, Y_2, Y_3)) + \psi_1^{(1)}(X, \varphi_3(Y_1, Y_2, Y_3)) - \varphi_3(\psi_2^{(1)}(X, Y_1), Y_2, Y_3) \\ - \psi_3^{(1)}(\varphi_2(X, Y_1), Y_2, Y_3) - \varphi_3(Y_1, \psi_2^{(1)}(X, Y_2), Y_3) - \psi_3^{(1)}(Y_1, \varphi_2(X, Y_2), Y_3) \\ - \varphi_3(Y_1, Y_2, \psi_2^{(1)}(X, Y_3)) - \psi_3^{(1)}(Y_1, Y_2, \psi_2^{(1)}(X, Y_3)) = 0, \\ \varphi_2(Y_1, \psi_3^{(1)}(Y_2, Y_3, Y_4)) + \psi_2^{(1)}(Y_1, \varphi_3(Y_2, Y_3, Y_4)) + \varphi_2(Y_2, \psi_3^{(1)}(Y_1, Y_3, Y_4)) \\ + \psi_2^{(1)}(Y_2, \varphi_3(Y_1, Y_3, Y_4)) + \varphi_2(Y_3, \psi_3^{(1)}(Y_1, Y_2, Y_4)) + \psi_2^{(1)}(Y_3, \varphi_3(Y_1, Y_2, Y_4)) \\ + \varphi_2(Y_4, \psi_3^{(1)}(Y_1, Y_2, Y_3)) + \psi_2^{(1)}(Y_4, \varphi_3(Y_1, Y_2, Y_3)) = 0.\end{aligned}\tag{6.10}$$

Using (6.2) these equations write

$$\varphi \circ_i \psi + \psi \circ_i \varphi = 0, \text{ with } i = 1, \dots, 4,\tag{6.11}$$

which is just equation (6.6) for $r = 1$.

Furthermore, the coefficient of t^2 obtained from (6.3) gives

$$\begin{aligned}
& \psi_1^{(1)}(\psi_1^{(1)}(X_1, X_2), X_3) + \psi_1^{(1)}(\psi_1^{(1)}(X_3, X_1), X_2) + \psi_1^{(1)}(\psi_1^{(1)}(X_2, X_3), X_1) = 0, \\
& \psi_2^{(1)}(\psi_1^{(1)}(X_1, X_2), Y) + \psi_2^{(1)}(\psi_2^{(1)}(X_2, Y), X_1) + \psi_2^{(1)}(\psi_2^{(1)}(Y, X_1), X_2) = 0, \\
& \psi_1^{(1)}(X, \psi_3^{(1)}(Y_1, Y_2, Y_3)) - \psi_3^{(1)}(\psi_2^{(1)}(X, Y_1), Y_2, Y_3) \\
& - \psi_3^{(1)}(Y_1, \psi_2^{(1)}(X, Y_2), Y_3) - \psi_3^{(1)}(Y_1, Y_2, \psi_2^{(1)}(X, Y_3)) = 0, \\
& \psi_2^{(1)}(Y_1, \psi_3^{(1)}(Y_2, Y_3, Y_4)) + \psi_2^{(1)}(Y_2, \psi_3^{(1)}(Y_1, Y_2, Y_4)) \\
& + \psi_2^{(1)}(Y_3, \psi_3^{(1)}(Y_1, Y_2, Y_4)) + \psi_2^{(1)}(Y_4, \psi_3^{(1)}(Y_1, Y_2, Y_3)) = 0,
\end{aligned} \tag{6.12}$$

which writes

$$\psi^{(1)} \circ_i \psi^{(1)} = 0, \text{ with } i = 1, \dots, 4. \tag{6.13}$$

Definition 6.4 Denote by

$$Z(A) = \{(\psi_1, \psi_2, \psi_3) : A \rightarrow \mathfrak{g}\},$$

where ψ_i ($i = 1, 2, 3$) satisfy (6.10) and (6.12). The vector space $Z(A)$ is called the infinitesimal deformation space of A .

6.4 Isomorphic deformations

Proposition 6.5 Let $(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \varphi) \in \mathcal{F}_{m,n}$ be an elementary Lie algebra of order 3. We consider a formal change of basis given by $\text{Id} + tf_0 \in GL(\mathfrak{g}_0 \otimes \mathbb{C}[[t]])$, $\text{Id} + tf_1 \in GL(\mathfrak{g}_1 \otimes \mathbb{C}[[t]])$. The isomorphic multiplication φ_t writes as the deformation

$$\varphi_t = \varphi + t\psi + \mathcal{O}(t^2),$$

where $\psi = (\psi_1, \psi_2, \psi_3)$ is given by

$$\begin{aligned}
\psi_1(X_1, X_2) &= \varphi_1(f_0(X_1), X_2) + \varphi_1(X_1, f_0(X_2)) - f_0(\varphi_1(X_1, X_2)), \\
\psi_2(X, Y) &= \varphi_2(f_0(X), Y) + \varphi_2(X, f_1(Y)) - f_1(\varphi_2(X, Y)), \\
\psi_3(Y_1, Y_2, Y_3) &= \varphi_3(f_1(Y_1), Y_2, Y_3) + \varphi_3(Y_1, f_1(Y_2), Y_3) + \varphi_3(Y_1, Y_2, f_0(Y_3)) - f_0(\varphi_3(Y_1, Y_2, Y_3)).
\end{aligned} \tag{6.14}$$

Proof. We put

$$\begin{aligned}
\tilde{X}_i &= (\text{Id} + tf_0)(X_i) = X_i + tf_0(X_i), \\
\tilde{Y}_j &= (\text{Id} + tf_1)(Y_j) = Y_j + tf_1(Y_j),
\end{aligned} \tag{6.15}$$

and we have

$$\begin{aligned}
\varphi_1(\tilde{X}_1, \tilde{X}_2) &= h_0^{-1} \varphi_1(h_0(X_1), h_0(X_2)), \\
\varphi_2(\tilde{X}_1, \tilde{Y}_2) &= h_1^{-1} \varphi_2(h_0(X_1), h_1(Y_2)), \\
\varphi_3(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) &= h_0^{-1} \varphi_3(g_1(Y_1), g_1(Y_2), g_1(Y_3)).
\end{aligned} \tag{6.16}$$

This can be written as

$$\begin{aligned}
\varphi_1(\tilde{X}_1, \tilde{X}_2) &= \varphi_1(X_1, X_2) + t\psi_1(X_1, X_2) + \mathcal{O}(t^2), \\
\varphi_2(\tilde{X}_1, \tilde{Y}_2) &= \varphi_2(X, Y) + t\psi_2(X, Y) + \mathcal{O}(t^2), \\
\varphi_3(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3) &= \varphi_3(Y_1, Y_2, Y_3) + t\psi_3(Y_1, Y_2, Y_3) + \mathcal{O}(t^2),
\end{aligned} \tag{6.17}$$

where, by a tedious but straightforward calculation, one has (6.14). QED

Definition 6.6 *An elementary Lie algebra of order 3 $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called rigid if all deformations of \mathfrak{g} are isomorphic to \mathfrak{g} .*

If \mathfrak{g} is rigid then \mathfrak{g}_0 is a rigid Lie algebra and the representation \mathfrak{g}_1 of \mathfrak{g}_0 is also rigid.

As an example of rigid Lie algebra of order 3 one has $\mathfrak{sl}(2) \oplus \text{ad } \mathfrak{sl}(2)$ with $\alpha = 1$ (see Theorem 3.1 for notations). Other examples are given in subsections 5.1 and 5.2. An example of non-rigid Lie algebra of order 3 is also exhibited in subsection 5.2. Finally, note that some rigidity properties of representations of $\mathfrak{sl}(2)$ can be found in [7].

Remark. Usually, rigidity is computed using cohomological methods with the Nijenhuis-Richardson theorem. This implies that we are able to define cohomology of algebra of order 3 with values in a \mathfrak{g} -module. For instance, it is easy to define 1 and 2 cochains and the corresponding cocycles and coboundaries space. For degree greater than 2, this is an open problem. For degree 2, we can quickly summarize the construction.

One denotes ψ (see Proposition 6.5) by $\delta_\varphi f$:

$$\begin{aligned}\psi_1(X_1, X_2) &= (\delta_{\varphi_1} f)(X_1, X_2), \\ \psi_2(X_1, Y_2) &= (\delta_{\varphi_2} f)(X_1, Y_2), \\ \psi_3(Y_1, Y_2, Y_3) &= (\delta_{\varphi_3} f)(Y_1, Y_2, Y_3)\end{aligned}\tag{6.18}$$

for any $X_i \in \mathfrak{g}_0$ and $Y_j \in \mathfrak{g}_1$. Let $Z(A) = Z^2(A)$ be the space of infinitesimal deformations (see Definition 6.3). Let $B^2(A)$ be the subspace of $Z^2(A)$ defined by

$$B^2(A) = \{\psi \in Z^2(A) : \psi = \delta_\varphi f\},$$

we obviously have $B^2(A) \subset Z^2(A)$. The theorem of Nijenhuis Richardson is written in this frame :
If $H^2 = Z^2/B^2 = \{0\}$, then elementary Lie algebra of order 3 \mathfrak{g} is rigid.

6.5 Some deformations on the Poincaré algebra of order 3

Let us consider the algebra (5.6). It leads to an explicit example of deformation. Let $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be the law defined by (5.6). The deformation $\varphi_t = (\varphi_{t1}, \varphi_{t2}, \varphi_{t3})$ is given by

$$\begin{aligned}\varphi_{t1}(L_{mn}, L_{pq}) &= \varphi_1(L_{mn}, L_{pq}), \\ \varphi_{t1}(L_{mn}, P_p) &= \varphi_1(L_{mn}, P_p), \\ \varphi_{t1}(P_m, P_p) &= -t^2 L_{mp}, \\ \varphi_{t2}(L_{mn}, V_{pq}) &= \varphi_2(L_{mn}, V_{pq}), \\ \varphi_{t2}(L_{mn}, V_p) &= \varphi_2(L_{mn}, V_p), \\ \varphi_{t2}(P_m, V_{pq}) &= t(\eta_{mp} V_q - \eta_{mq} V_p), \\ \varphi_{t2}(P_m, V_p) &= t V_{mp}, \\ \varphi_{t3}(V_{mn}, V_{pq}, V_{rs}) &= t((\eta_{mp} \eta_{nq} - \eta_{mq} \eta_{np}) L_{rs} + (\eta_{mr} \eta_{ns} - \eta_{ms} \eta_{nr}) L_{pq} + (\eta_{pr} \eta_{qs} - \eta_{ps} \eta_{qr}) L_{mn}), \\ \varphi_{t3}(V_{mn}, V_{pq}, V_r) &= \varphi_3(V_{mn}, V_{pq}, V_r), \\ \varphi_{t3}(J_{mn}, P_p, P_r) &= t \eta_{pr} L_{mn}, \\ \varphi_{t3}(V_m, P_p, P_r) &= \varphi_3(V_m, V_p, V_r).\end{aligned}$$

and we obtain (5.4).

Acknowledgment: We would like to thank M. J. Slupinski for help in proving Lemma 4.1.

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